Part III Essay
Easter 2002

The Dynkin Diagrams of Rational Double Points

Benjamin Friedrich
Trinity College

Home address:
Kleiner Warnowdamm 10
18109 Rostock
Germany
bf216@hermes.cam.ac.uk

15 May 2002

I declare that this essay is work done as part of the Part III Examination. It is the result of my own work, and except where stated otherwise, includes nothing which was performed in collaboration. No part of this essay has been submitted for a degree or any such qualification.

Benjamin Friedrich

Part III Essay
Easter 2002

The Dynkin Diagrams of Rational Double Points

Abstract

Rational double points are the simplest surface singularities. In this essay we will be mainly concerned with the geometry of the exceptional set corresponding to the resolution of a rational double point. We will derive the classification of rational double points in terms of Dynkin diagrams.

Contents

1 Introduction

Rational double points[1] are the simplest surface singularities and were first studied by Du Val [10]. One may think of them as neglible singularities. They

[1] In the literature, they are also called Du Val singularities, Kleinian singularities or simple critical points.

play an important rôle in the classification of surfaces and occur in the theory of simultaneous resolutions of singularities.
In this essay we will be mainly concerned with the geometry of the exceptional set corresponding to a resolution of a rational double point. We will derive the classification of rational double points in terms of Dynkin diagrams. It should be noted, that the proof of this classification is rather lengthy. However, the author was unable to find the complete proof in a single source and therefore decided to present it in full detail. Most ideas are taken from two papers of Artin [1], [2], balanced with a slightly different approach in Reid's draft [27]. Further parts of the argument are taken from Durfee [9], Mumford [24] and Brieskorn [4]. The second article by Pinkham in [8] treats the topic very nicely, although some difficult steps are omitted.
Finally, we will find a connection between the most simple objects in different fields of mathematics: Rational double points are linked with Platonic solids and simple Lie groups.

2 Basic facts on surface singularities

2.1 Definitions

We want to study *surface singularities* (X, x); here X is a normal, two-dimensional, projective variety over $\mathbb{C}$ which is non-singular, except maybe at $x \in X$. Two singularities are isomorphic, if there exist open neighbourhoods of the singular points which are isomorphic.
A *resolution* of (X, x) is a birational, proper and surjective morphism

$$\pi : \widetilde{X} \to X$$

where $\widetilde{X}$ is a non-singular projective variety over $\mathbb{C}$.
See section 9 for an example.
It is an important and difficult theorem, that resolutions always exist; for a general discussion we refer to [22], [21].

Immediate properties of the exceptional set The *exceptional set* $E := \pi^{-1}(x)$ is compact (since X is proper) and one-dimensional (since π is birational). Moreover it is connected by Zariski's connectedness theorem A.5. Therefore E is a bunch of irreducible projective curves

$$E = \bigcup_{i=1}^{n} E_i.$$

We say that a surface singularity is

rational, if for a resolution

$$\pi : \widetilde{X} \to X$$

the first higher direct image sheaf of $\widetilde{X}$'s structure sheaf vanishes

$$R^1\pi_*\mathcal{O}_{\widetilde{X}} = 0, \qquad (1)$$

and

a double point, if the local ring $\mathcal{O}_{X,x}$ has multiplicity two, i.e. the leading coefficient of its Hilbert-Samuel polynomial is two ([25] III §23, [33] vol. 2, VIII §10).

Remarks:

1. (a) The definition of a rational singularity is independent of the chosen resolution: Since $R^1\pi_*\mathcal{O}_X$ is a coherent sheaf ([17] III.8.8.(b)) concentrated on x, all we are interested in is $h^0(X, R^1\pi_*\mathcal{O}_{\widetilde{X}})$. However, we will see soon in section 2.2 that

$$p_a(X) - p_a(\widetilde{X}) = h^0(X, R^1\pi_*\mathcal{O}_{\widetilde{X}})$$

and the arithmetic genus of a **non-singular** projective variety is a birational invariant ([17] V.5.6).

   (b) The condition (1) may appear opaque at a first glance, but will hopefully become more transparent in the sequel. For example, it implies that the E_i are rational curves.

2. Since we are in the normal case, the condition for a double point means, that two general curves on X through x have local intersection number two at x ([27] 4.6). If X is a hypersurface $f^{-1}(0)$, yet another way to state this condition is

$$f \in m_x^2 \text{ and } f \notin m_x^3,$$

where m_x is the ideal of functions vanishing at x ([7] (7.48)).

2.2 A first consequence of the rationality condition (1)

Let us mention a simple consequence of the rationality condition (1).

Proposition 2.1 *Let $\pi : \widetilde{X} \to X$ be a resolution as above. If*

$$R^1\pi_*\mathcal{O}_{\widetilde{X}} = 0$$

then

$$p_a(X) = p_a(\widetilde{X}).$$

We need a lemma.

Lemma 2.2 *For any resolution $\pi : \widetilde{X} \to X$, we have*

$$\pi_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X.$$

Proof ([17] p. 280): Since the question is local on X, we can assume X is affine, say $X = \operatorname{Spec} A$. By [17] II.5.8.(b), $\pi_*\mathcal{O}_{\widetilde{X}}$ is a coherent sheaf of $\mathcal{O}_X$-algebras, hence $B := H^0(X, \pi_*\mathcal{O}_{\widetilde{X}})$ is a finitely generated A-module. But A and B are integral domains with the same quotient field (since π is birational) and A is algebraically closed (since X is normal), thus $B = A$, and $\pi_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$. □

Proof of the proposition ([17] Ex. III.8.1): Let

$$0 \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \longrightarrow \dots \qquad (2)$$

be an injective resolution for $\mathcal{O}_{\widetilde{X}}$. We have not only $R^1\pi_*\mathcal{O}_{\widetilde{X}} = 0$, but $R^i\pi_*\mathcal{O}_{\widetilde{X}} = 0$ for $i \geq 1$, because the fibers of π have dimension ≤ 1 (A.7). Therefore, by applying π_* to (2), we obtain again an exact sequence

$$0 \longrightarrow \pi_*\mathcal{O}_{\widetilde{X}} \longrightarrow \pi_*I^0 \xrightarrow{\pi_*d^0} \pi_*I^1 \xrightarrow{\pi_*d^1} \pi_*I^2 \longrightarrow \dots . \qquad (3)$$

Since injectives are flasque ([17] III.2.4), direct images of flasque sheaves are flasque, and flasque sheaves are acyclic for the global section functor ([17] III.2.5), we see that (3) is an acyclic resolution for $\mathcal{O}_X = \pi_*\mathcal{O}_{\widetilde{X}}$. Thus

$$\begin{aligned} H^i(X, \mathcal{O}_X) &= \frac{\ker H^0(X, \pi_*d^i)}{\operatorname{im} H^0(X, \pi_*d^{i-1})} \\ &= \frac{\ker H^0(\widetilde{X}, d^i)}{\operatorname{im} H^0(\widetilde{X}, d^{i-1})} \\ &= H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \end{aligned}$$

and our claim follows. □

Remark: This proof is just a degenerated case of the *Leray spectral sequence*

$$\begin{aligned} &E_2^{p,q} = H^p(X, R^q\pi_*\mathcal{O}_{\widetilde{X}}) \\ &\Rightarrow \\ &E_\infty^{p,q} = H^{p+q}(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \end{aligned}$$

(cf. [14] II.4.17.1, [32] V; for general information on spectral sequences cf. [23], [28] II §4), which takes in our setting the simple form (theorem A.6 and A.7)

$$\begin{array}{rcccc} & 0 & 0 & 0 & 0 \\ E_2^{p,q}: & H^0(X, R^1\pi_*\mathcal{O}_{\widetilde{X}}) & ? & ? & 0 \\ & H^0(X, \mathcal{O}_X) & H^1(X, \mathcal{O}_X) & H^2(X, \mathcal{O}_X) & 0 \\ \hline \end{array}$$

$$H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = E_\infty^{i,0} = E_2^{i,0} = H^i(X, \mathcal{O}_X) \text{ for } i = 0, 1,$$
$$H^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = E_\infty^{2,0} = E_2^{2,0}/E_2^{0,1} = H^2(X, \mathcal{O}_X)/H^0(X, R^1\pi_*\mathcal{O}_{\widetilde{X}}).$$

From this we get

$$p_a(X) - p_a(\widetilde{X}) = h^0(X, R^1\pi_*\mathcal{O}_{\widetilde{X}})$$

and our proposition (and its converse!) follow at once.

2.3 Further properties of the exceptional set E

It will be a great technical convenience to work with *good resolutions*; for them we require that

1. all E_i are non-singular,

2. $E_i \cap E_j \cap E_k = \varnothing$ for mutually distinct i, j, k,

3. the intersection of E_i and E_j is transverse for $i \neq j$.

Any resolution $\pi : \widetilde{X} \to X$ of a **surface** X can be brought in such a nice form by successively blowing up points of $\widetilde{X}$ (cf. again [22], [21], and also [17] V.3.8, V.3.9).
In the following we do always assume that $\pi : \widetilde{X} \to X$ is good.
A fundamental fact about good resolutions is the following

Proposition 2.3 ([24] p. 6) *The* intersection matrix of the resolution $(E_i \cdot E_j)_{i,j=1\ldots n}$ *is negative definite.*

Proof: We take a meromorphic function $f \in k(X)$ with $f(x) = 0$ and define two effective divisors

$$H_0 := f^{-1}(0) \text{ and } H_\infty := f^{-1}(\infty).$$

Denote the proper transform of H_i with $\widetilde{H}_i$ for $i = 0, \infty$ respectively. Then we have a linear equivalence of divisors

$$\widetilde{H}_\infty \sim \widetilde{H}_0 + \sum_{i=1}^{n} m_i E_i$$

where $m_i = \mathrm{ord}_{E_i} f \circ \pi > 0$.
It suffices to prove that the matrix

$$M := (m_i E_i \cdot m_j E_j)_{i,j=1,\ldots,n}$$

is negative definite. Now we have $M_{i,j} \geq 0$ if $i \neq j$ (since the E_i are irreducible) and

$$\begin{aligned} \sum_{i=1}^{n} M_{i,j} &= \sum_{i=1}^{n} (m_i E_i \cdot m_j E_j) \\ &= (\widetilde{H}_\infty - \widetilde{H}_0) \cdot m_j E_j \\ &= 0 - \widetilde{H}_0 \cdot m_j E_j \leq 0. \end{aligned}$$

This implies that M is negative semi-definite:

$$\begin{aligned} \sum_{i,j=1}^{n} a_i a_j M_{i,j} &= \sum_{i=1}^{n} a_i^2 M_{i,i} + 2 \sum_{\substack{i,j=1 \\ i<j}}^{n} a_i a_j M_{i,j} \\ &= \sum_{j=1}^{n} \underbrace{\left(\sum_{i=1}^{n} M_{i,j} \right)}_{\leq 0} a_j^2 - \sum_{\substack{i,j=1 \\ i<j}}^{n} \underbrace{M_{i,j} (a_i - a_j)^2}_{\geq 0} \leq 0. \end{aligned} \tag{4}$$

To show definiteness, we note that $\widetilde{H}_0$ must pass through some E_i, hence

$$\sum_{i=1}^{n} M_{i,j_0} < 0 \text{ for some } j_0.$$

Suppose we have equality in (4). Then $a_{j_0} = 0$. Furthermore, we get $a_i = a_j$ if $M_{i,j} > 0$, or inductively $a_i = a_j$ if E_i and E_j are connected in E. But E is connected, hence $a_i = 0$ for $i = 1, \dots, n$ in this case. □

In the proof of proposition 2.3, we encountered an effective *exceptional divisor* (i.e. an divisor supported on E)

$$Z := \widetilde{H}_\infty - \widetilde{H}_0 = \sum_{i=1}^{n} m_i E_i > 0$$

which had the note-worthy property

$$Z \cdot E_i \leq 0 \text{ for } i = 1, \dots, n. \tag{5}$$

Since E is connected, any exceptional divisor Z with this property (5) must satisfy $Z \geq E$, by the arguments used in that proof. If two exceptional divisors $Z^1 = \sum_{i=1}^{n} r_i^1 E_i > 0$ and $Z^2 = \sum_{i=1}^{n} r_i^2 E_i > 0$ both satisfy (5) then obviously so does $Z := \min(Z^1, Z^2) = \sum_{i=1}^{n} r_i E_i > 0$ where $r_i := \min(r_i^1, r_i^2)$: $Z \cdot E_i \leq Z^j \cdot E_i \leq 0$ whenever $r_i = r_i^j$. Hence there exists a minimal positive exceptional divisor, called the *numerical divisor* Z_{num} ([27] 4.5) (also called *fundamental divisor* [2]), for which (5) holds.
This divisor Z_{num} will provide a useful tool to describe the exceptional set of a rational singularity.

3 The geometry of the exceptional set E of a resolution of a rational singularity

Throughout this section, we assume that $\pi : \widetilde{X} \to X$ is good resolution of a rational singularity (X, x) and E its exceptional set.
We will prove that the E_i are rational curves $E_i \cong \mathbb{P}^1$. Moreover, we will be able to read off the multiplicity of (X, x) as the self-intersection-number $-(Z_{\text{num}})^2$.
The idea is to study fatter and fatter infinitesimal neighbourhoods of E in order to examine the embedding of E in $\widetilde{X}$.
We will identify an exceptional divisor $Z = \sum_{i=1}^{n} r_i E_i$ with its associated *positive cycle*: this is the, generally **non-reduced**, scheme $(\text{Supp } Z, \mathcal{O}_Z)$. Recall that $\mathcal{O}_Z = \text{coker}(\mathcal{O}_{\widetilde{X}}(-Z) \to \mathcal{O}_{\widetilde{X}})$, i.e. $(\text{Supp } Z, \mathcal{O}_Z)$ is the subscheme of $\widetilde{X}$ defined by the coherent sheaf of ideals on $\widetilde{X}$ whose sections on an open $U \subset \widetilde{X}$ are the rational functions $f \in \Gamma(U, \mathcal{O}_{\widetilde{X}})$ which have zeros of order at least r_i along E_i for all i with $E_i \cap U \neq \varnothing$ [1]. Note $\text{Supp } Z = \bigcup_{r_i > 0} E_i$.

3.1 The exceptional curves E_i are rational

Theorem 3.1 ([2] prop. 1, [4] Lemma 1.3) *The exceptional set of a good resolution of a rational singularity consists of rational projective curves $E_i \cong \mathbb{P}^1$.*

Proof ([2], [1]):
The proof relies on Grothendieck's theorem on formal functions A.3, which takes in our case the form

$$0 = \left(\left(R^1\pi_*\mathcal{O}_{\tilde{X}}\right)_x\right)^\wedge \cong \varprojlim_{k=1}^{\infty} H^1\left(E, \mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/{m_x}^k\right)$$

where $m_x \subset \mathcal{O}_{X,x}$ is the maximal ideal corresponding to x and completion is taken with respect to the m_x-adic topology.
(We will see later in lemma 3.8, that $\mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/{m_x}^k = \mathcal{O}_{kZ_{\text{num}}}$.)
Since E is one-dimensional, the natural map

$$\mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/{m_x}^{k+1} \twoheadrightarrow \mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/{m_x}^k$$

induces a surjection on cohomology

$$H^1\left(E, \mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/{m_x}^{k+1}\right) \twoheadrightarrow H^1\left(E, \mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/{m_x}^k\right)$$

by a vanishing theorem of Grothendieck A.6. Thus we see

$$H^1\left(E, \mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/{m_x}^{k+1}\right) = 0 \text{ for all } k \in \mathbb{N}.$$

We denote the sheaf of ideals of functions vanishing at x by $\mathfrak{m}_x$. Clearly every function in $\mathfrak{m}_x \cdot \mathcal{O}_{\tilde{X}}$ vanishes on E; hence for every positive cycle Z we can find an integer k such that every function in ${\mathfrak{m}_x}^k \cdot \mathcal{O}_{\tilde{X}}$ vanishes on Z. We now have

$$\mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/{m_x}^k \twoheadrightarrow \mathcal{O}_Z$$

and (A.6)

$$0 = H^1\left(E, \mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}/{m_x}^{k+1}\right) \twoheadrightarrow H^1(E, \mathcal{O}_Z).$$

In particular $H^1(E, \mathcal{O}_{E_i}) = 0$ for $i = 1, \ldots, n$, from which we conclude $p_a(E_i) = 0$, i.e. $E_i \cong \mathbb{P}^1$. □

Corollary 3.2 *In the proof of theorem 3.1 we have just seen $H^1(E, \mathcal{O}_Z) = 0$ for every positive cycle Z.*

We can make a more precise statement for the numerical divisor Z_{num}.

Corollary 3.3 ([2] thm. 3) *With the assumptions of the theorem we have*

$$p_a(Z_{\text{num}}) = 0.$$

Proof ([27] 3.11): The statement follows immediately from corollary 3.3 and the general fact $\boxed{h^0(E, \mathcal{O}_{Z_{\text{num}}}) = 1}$. This can be proved easily by induction: We know $h^0(E) = 1$. Assume $h^0(Y) = 1$ for a positive cycle $E \le Y \lneq Z_{\text{num}}$. We have

$$Y \cdot E_i > 0 \text{ for some } i,$$

or equivalently $\deg_{E_i} \mathcal{O}_E(-Y) \le -1$, by the very definition of Z_{num}. Certainly $Y + E_i \le Z_{\text{num}}$ in this situation. From

$$0 = H^0(E_i, \mathcal{O}_{E_i}(-Y)) \to H^0(E, \mathcal{O}_{Y+E_i}) \to H^0(E, \mathcal{O}_Y)$$

we conclude $h^0(E, \mathcal{O}_{Y+E_i}) = 1$. □

3.2 A criterion for rationality

Theorem 3.4 ([2], thm. 3)
Conversely, if we have

$$p_a(Z_{\text{num}}) = 0$$

for the numerical cycle of a good resolution of a singularity (X, x), then (X, x) is rational.

We need the following lemma.

Lemma 3.5 *Let $Z = \sum_{i=1}^{n} r_i E_i$ be a positive cycle with the property that $p_a(Y) \leq 0$ for all positive cycles $Y \leq Z$. Then $H^1(E, \mathcal{O}_Z) = 0$.*

Proof: In particular $p_a(E_i) = 0$ for all i with $r_i \geq 1$, i.e. $E_i \cong \mathbb{P}^1$. We use induction on $\sum_{i=1}^{n} r_i$. Assume $H^1(E, \mathcal{O}_Z) \neq 0$. Let $Z_i := Z - E_i$ for $r_i \geq 1$. By induction hypothesis

$$H^1(E, \mathcal{O}_{Z_i}) = 0,$$

hence for the kernel M

$$0 \to M \to \mathcal{O}_Z \to \mathcal{O}_{Z_i} \to 0$$

we get (A.6)

$$H^1(E, M) \to H^1(E, \mathcal{O}_Z) \to 0,$$

i.e. $H^1(E, M) \neq 0$. By the snake-lemma

$$\begin{array}{ccccccccc}
 & & & & 0 & \longrightarrow & M & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\tilde{X}}(-Z) & \longrightarrow & \mathcal{O}_{\tilde{X}} & \longrightarrow & \mathcal{O}_Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\tilde{X}}(-Z_i) & \longrightarrow & \mathcal{O}_{\tilde{X}} & \longrightarrow & \mathcal{O}_{Z_i} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{\tilde{X}}(-Z_i) \otimes \mathcal{O}_{E_i} & \longrightarrow & 0 & & & &
\end{array}$$

we deduce $M \cong \mathcal{O}_{\tilde{X}}(-Z_i) \otimes \mathcal{O}_{E_i}$. Hence we can write

$$0 \neq H^1(E, \mathcal{O}_{\tilde{X}}(-Z_i) \otimes \mathcal{O}_{E_i}) = H^1(E_i, \mathcal{O}_{\tilde{X}}(-Z_i) \otimes \mathcal{O}_{E_i}).$$

But E_i is just $\mathbb{P}^1$, thus ([17] III.5.1)

$$\deg \mathcal{O}_{\tilde{X}}(-Z_i) \otimes \mathcal{O}_{E_i} \leq -2.$$

On the other hand

$$\deg \mathcal{O}_{\tilde{X}}(-Z_i) \otimes \mathcal{O}_{E_i} = -Z_i \cdot E_i$$

and by the adjunction formula A.2 we get

$$Z \cdot E_i = (Z_i + E_i) \cdot E_i \geq 2 + {E_i}^2 = -K \cdot E_i.$$

Summing up $(Z + K) \cdot E_i \geq 0$ yields with the adjunction formula A.2

$$2p_a(Z) - 2 = (Z + K) \cdot Z \geq 0,$$

i.e. $p_a(Z) \geq 1$, a contradiction. □

Proof of the theorem ([2] prop. 1, thm. 3): We have seen in the proof of theorem 3.1, that by Grothendieck's theorem on formal functions (A.3)

$$0 = \left(\left(R^1\pi_*\mathcal{O}_{\widetilde{X}}\right)_x\right)\hat{} \cong \varprojlim_{k=1}^{\infty} H^1(E, \mathcal{O}_{kZ_{\text{num}}}),$$

hence it is sufficient to prove

$$H^1(E, \mathcal{O}_{kZ_{\text{num}}}) = 0 \text{ for all } k.$$

We already know $H^1(E, \mathcal{O}_{Z_{\text{num}}}) = 0$ (since $p_a(Z_{\text{num}}) = 0$ and $h^0(E, \mathcal{O}_{Z_{\text{num}}}) = 1$). From the surjection (A.6)

$$H^1(E, \mathcal{O}_{Z_{\text{num}}}) \twoheadrightarrow H^1(E, \mathcal{O}_{E_i})$$

we find that $p_a(E_i) = 0$, i.e. $E_i \cong \mathbb{P}^1$.
By the lemma, it is enough to show $p_a(Y) \leq 0$ for all positive cycles Y. Let $Y_1 := Y$ and define Y_{n+1} inductively as follows

1. if $Y_n \geq Z_{\text{num}}$, then $Y_{n+1} := Y_n - Z_{\text{num}} \geq 0$.
2. if $Y_n \not\geq Z_{\text{num}}$, then $Y_n \cdot E_i > 0$ for some i by the definition of Z_{num}. Choose such an i with smallest possible multiplicity in Y_n and set $Y_{n+1} := Y_n + E_i$.

Stop when $Y_n = 0$. We use the equation ([17] Ex. V.1.3)

$$p_a(Z_1 + Z_2) = p_a(Z_1) + p_a(Z_2) + Z_1 \cdot Z_1 - 1 \tag{6}$$

to calculate the arithmetic genus:
In case 1:

$$\begin{aligned} p_a(Y_n) &= p_a(Y_{n+1} + Z_{\text{num}}) \\ &= p_a(Y_{n+1}) + p_a(Z_{\text{num}}) + Y_{n+1} \cdot Z_{\text{num}} - 1 \\ &\leq p_a(Y_{n+1}) - 1 \end{aligned}$$

In case 2:

$$\begin{aligned} p_a(Y_{n+1}) &= p_a(Y_n) + p_a(E_i) + Y_n \cdot E_i - 1 \\ &\geq p_a(Y_n). \end{aligned}$$

Steps of type (2) cannot be repeated infinitely often without reaching a stage where $Y_n \geq Z_{\text{num}}$. Using equation (6) once again, we see that $p_a(Y)$ is a quadratic form in the coefficients s_i of $Y = \sum_{i=1}^{n} s_i E_i$ whose quadratic term is

$\frac{1}{2}\sum_{i,j=1}^{n} s_i s_j E_i \cdot E_j$. But the matrix $(E_i \cdot E_j)_{i,j=1,\ldots,n}$ is negative definite, hence $p_a(Y)$ is bounded above. Consequently, there can be only a finite number of steps and the algorithm must terminate with $Y_n = 0$. Then $Y_{n-1} = Z_{\text{num}}$ and we have

$$p_a(Y_1) \leq \cdots \leq p_a(Y_{n-1}) = 0.$$

□

3.3 Invertible sheaves on a positive cycle Z

It is a natural question to ask what are the invertible sheaves on a positive cycle Z, i.e. what is Pic Z?
In our particular case the answer is quite simple. It will provide an important tool in exploring the geometry of E further.
For any invertible sheaf $\mathcal{F}$ on the positive cycle $Z \geq E$, we can define its *multidegree*

$$\deg_Z : \text{Pic } Z \to \mathbb{Z}^n$$

via the composite maps

$$\text{Pic } Z \to \text{Pic } E_i \stackrel{\deg}{\longrightarrow} \mathbb{Z} \text{ for } i = 1, \ldots, n.$$

Using *local transverse cuts* it is easily seen that this map is surjective $\deg_Z \text{Pic } Z \twoheadrightarrow \mathbb{Z}^n$: Choose a general point p on any E_i and construct a Cartier divisor $\{(U_j, f_j)\}$ with support p and degree 1 on E_i whose local equation $s \in \mathcal{O}_{Z,p}$ restricts to a local equation of p in $\mathcal{O}_{E_i,p}$. This gives

$$\deg_Z\{(U_j, f_j)\} = (0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0).$$

In fact, we will prove that $\deg_Z \text{Pic } Z \twoheadrightarrow \mathbb{Z}^n$ is an isomorphism.
It is a well-known fact that ([17] Ex. III.4.5)

$$\text{Pic } Z = H^1(E, \mathcal{O}_Z^*).$$

(One may think of a Čhech-1-cocycle $\{(U_i \cap U_j, g_{i,j})\} \in H^1(E, \mathcal{O}_Z^*)$ as a set of transition functions $g_{i,j} : \mathcal{O}_{U_i \cap U_j} \stackrel{\sim}{\longrightarrow} \mathcal{O}_{U_i \cap U_j}$ which define a line bundle on Z.)
In the reduced case $Z = E$, it is easy to see what $H^1(E, \mathcal{O}_E^*)$ is, if we allow transcendental methods. Let $\circ_h$ denote the functor form the category of schemes of finite type over $\mathbb{C}$ to the category of complex analytic spaces. (cf. [17] B and section 6).
Since E is projective over $\mathbb{C}$, a theorem by Serre ([17] B.2.1) tells us that

$$H^i(E, \mathcal{F}) \cong H^i(E_h, \mathcal{F}_h)$$

for every coherent sheaf $\mathcal{F}$ on E. The exponential sequence ([17] V.5)

$$0 \to \mathbb{Z} \to \mathcal{O}_{E_h} \stackrel{\exp 2\pi i}{\longrightarrow} \mathcal{O}_{E_h}^* \to 0$$

yields (corollary 3.2 and theorem A.6)

$$0 \to H^1(E_h, \mathcal{O}_{E_h}^*) \to H^2(E_h, \mathbb{Z}) \to 0,$$

i.e.

$$\text{Pic } E \cong H^2(E_h, \mathbb{Z}).$$

As we have already seen, E is built up out of n spheres $S^2 \cong \mathbb{P}^1$, which intersect each other transversely. Hence by the Mayer-Vietoris sequence for, say singular cohomology

$$H^2(E_h, \mathbb{Z}) \cong \mathbb{Z}^n.$$

(We will see later, that E has the homotopy type of a bouquet of n spheres $E \simeq (S^2)^{\vee n}$.) Therefore, we obtain

$$\text{Pic } E \cong \mathbb{Z}^n.$$

Unfortunately, there is no analogue of the exponential sequence in the non-reduced case. Instead, we need the following proposition by Artin, whose proof uses a "first order exponential".

Proposition 3.6 ([1] lemma 1.4) *We have*

$$H^1(E, \mathcal{O}_Z) \cong H^1(E, \mathcal{O}_E) \cong \mathbb{Z}^n$$

for every positive cycle $Z \geq E$.

Proof: We will proceed by induction: The case $Z = E$ is trivial, so assume the proposition holds for $Z' = Z - E_i \geq E$. We fix our notation for the following kernels

$$\begin{aligned}
0 \to \mathcal{N} \to \mathcal{O}_Z \to \mathcal{O}_E \to 0, \\
0 \to \mathcal{M} \to \mathcal{O}_Z^* \to \mathcal{O}_E^* \to 0, \\
0 \to \mathcal{N}' \to \mathcal{O}_{Z'} \to \mathcal{O}_E \to 0, \\
0 \to \mathcal{M}' \to \mathcal{O}_{Z'}^* \to \mathcal{O}_E^* \to 0, \\
0 \to \mathcal{J} \to \mathcal{O}_Z \to \mathcal{O}_{Z'} \to 0 \qquad \text{and} \\
0 \to \mathcal{K} \to \mathcal{O}_Z^* \to \mathcal{O}_{Z'}^* \to 0.
\end{aligned}$$

By A.6, it suffices to prove $H^1(E, \mathcal{M}) = 0$.
Note that $H^0(E, \mathcal{O}_E) = \mathbb{C}$ (and also $H^0(E, \mathcal{O}_E^*) = \mathbb{C}^*$), since E is connected. In particular, we get a surjection

$$H^0(E, \mathcal{O}_Z) \twoheadrightarrow H^0(E, \mathcal{O}_E),$$

which implies (corollary 3.2)

$$H^1(E, \mathcal{N}) = 0.$$

Similarly, $H^1(E, \mathcal{M}') = 0$ (using the induction hypothesis). Now, these kernels are linked by the short exact sequences

$$\begin{aligned}
0 \to \mathcal{J} \to \mathcal{N} \to \mathcal{N}' \to 0 \\
0 \to \mathcal{K} \to \mathcal{M} \to \mathcal{M}' \to 0
\end{aligned}$$

and we obtain

$$H^0(E,\mathcal{N}') \xrightarrow{\delta} H^1(E,\mathcal{J}) \to 0$$
$$H^0(E,\mathcal{M}') \xrightarrow{\delta'} H^1(E,\mathcal{K}) \to H^1(E,\mathcal{M}) \to 0.$$

Because of $\mathcal{J} \cdot \mathcal{N} = 0$ (thus $\mathcal{J}^2 = 0$), we have an isomorphism

$$\epsilon : \mathcal{J} \xrightarrow{\sim} \mathcal{K}$$

via

$$s \in \Gamma(U,\mathcal{J}) \mapsto 1 + s \in \Gamma(U,\mathcal{K}),$$

hence $H^1(E,\mathcal{J}) \cong H^1(E,\mathcal{K})$. Analogously, we have a bijection (not a morphism, in general!)

$$\begin{array}{rcl} \epsilon' : H^0(E,\mathcal{N}') & \to & H^0(E,\mathcal{M}') \\ s' & \mapsto & 1 + s'. \end{array}$$

Therefore, it suffices to show that the following diagram commutes

$$\begin{array}{ccc} H^0(E,\mathcal{N}') & \xrightarrow{\delta} & H^1(E,\mathcal{J}) \\ {\scriptstyle \epsilon'}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ H^0(E,\mathcal{M}') & \xrightarrow{\delta'} & H^1(E,\mathcal{K}). \end{array}$$

Pick an element $s' \in H^0(E,\mathcal{N}')$ and choose an open covering $\{U_i\}$ of E such that s' can be lifted to $s_i \in \Gamma(U_i,\mathcal{N})$. Now we can write $\delta(s')$ as the Čech-1-cocycle

$$\{(U_i \cap U_j, s_i - s_j)\} \in H^1(E,\mathcal{J})$$

and get

$$\epsilon(\delta(s')) = \{(U_i \cap U_j, 1 + s_i - s_j)\} \in H^1(E,\mathcal{K}).$$

In the same way, we can lift $\epsilon'(s') = 1 + s'$ to $1 + s_i \in \Gamma(U_i,\mathcal{M})$ and obtain

$$\epsilon'(\delta'(s')) = \{(U_i \cap U_j, \frac{1+s_i}{1+s_j})\} \in H^1(E,\mathcal{K}).$$

We use $\mathcal{J} \cdot \mathcal{N}$ in order to show

$$\epsilon(\delta(s')) = \epsilon'(\delta'(s')).$$

Since

$$s_i - s_j \in \Gamma(U_i \cap U_j, \mathcal{J})$$

we get

$$s_j(s_i - s_j) = 0,$$

hence

$$(1 + s_j)(1 + s_i - s_j) = 1 + s_i,$$

i.e.

$$1 + s_i - s_j = \frac{1+s_i}{1+s_j}.$$

This finishes the proof. □

3.4 The multiplicity of a rational singularity

The following theorem is the main result of this section.

Theorem 3.7 ([2] cor. 6) *The multiplicity of the rational singularity* (X, x) *is equal to the negative of the self-intersection-number of the numerical cycle* $-(Z_{\text{num}})^2$.

For the proof, we need two lemmas, which are interesting in their own rights.

Lemma 3.8 ([2] thm. 4) *We have*

$$\mathfrak{m}_x \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-Z_{\text{num}}).$$

Proof ([2], [27] 4.17):
The inclusion $\mathfrak{m}_x \cdot \mathcal{O}_{\widetilde{X}} \subseteq \mathcal{O}_{\widetilde{X}}(-Z_{\text{num}})$ is easy [2]: For $f \in \Gamma(U, \mathfrak{m}_x \cdot \mathcal{O}_{\widetilde{X}})$ we can split the principal divisor (f) in a part Z supported on E and a part D, which does not involve any of the E_i at all: $(f) = Z + D$. Obviously $Z > 0$. We have

$$(f) \cdot E_i = 0 \text{ and } D \cdot E_i \geq 0 \text{ for } i = 1, \ldots, n,$$

since f is regular in a neighbourhood of E. Thus $Z \cdot E_i \leq 0$ for $i = 1, \ldots, n$, that is $Z \geq Z_{\text{num}}$. Hence $f \in \Gamma(U, \mathcal{O}_{\widetilde{X}}(-Z_{\text{num}}))$.
For the other inclusion we have to show that for each point $p \in E$ there exists a local section f of $\mathfrak{m}_x \cdot \mathcal{O}_{\widetilde{X}}$ such that $(f)_{|U} = Z_{\text{num}|U}$ for a neighbourhood U of p ([27] 4.17).
Let X' be an affine neighbourhood of $x \in X$ and set $\widetilde{X}' := \widetilde{X} \times_X X' = \pi^{-1}(X')$.
We will write for short $\mathcal{J} := \mathcal{O}_{\widetilde{X}'}(-Z_{\text{num}})$.
We can construct a divisor A on Z_{num} as a sum of local transverse cuts such that $p \notin \text{Supp}A$ and $\deg_{Z_{\text{num}}} A = \deg_{Z_{\text{num}}} \mathcal{J}_{|Z_{\text{num}}}$. The crucial point is, that proposition 3.6 implies now $\mathcal{O}_{Z_{\text{num}}}(A) \cong \mathcal{J}_{|Z_{\text{num}}}$. Hence there exists a section $s \in H^0(E, \mathcal{J}_{|Z_{\text{num}}})$ which does not vanish at p.
To finish the proof, all we have to do is to lift s to a section on $\widetilde{X}'$.
From the short exact sequence

$$0 \to \mathcal{J}^{\otimes 2} \to \mathcal{J} \to \mathcal{J}_{|Z_{\text{num}}} \to 0$$

(obtained from $0 \to \mathcal{J} \to \mathcal{O}_{\widetilde{X}'} \to \mathcal{O}_{Z_{\text{num}}} \to 0$ by tensoring with $\mathcal{J}$) we get

$$H^0(\widetilde{X}', \mathcal{J}) \to H^0(E, \mathcal{J}_{|Z_{\text{num}}}) \to H^1(\widetilde{X}', \mathcal{J}^{\otimes 2}).$$

So it suffices to proof $H^1(\widetilde{X}', \mathcal{J}^{\otimes 2}) = 0$. We will prove more generally:

Useful fact: $H^1(\widetilde{X}', \mathcal{J}^{\otimes k}) = 0$ for all $k \in \mathbb{N}$.
Proof of the useful fact: Since X' is affine

$$H^1(\widetilde{X}', \mathcal{J}^{\otimes k}) = H^0(X', R^1\pi_*\mathcal{J}^{\otimes k})$$

by [17] III.8.5.
The sheaf $R^1\pi_*\mathcal{J}^{\otimes k}$ is concentrated in x; hence it is enough to prove $\left(R^1\pi_*\mathcal{J}^{\otimes k}\right)_x = 0$. By Grothendieck's theorem on formal functions (A.3)

$$\left((R^1\pi_*\mathcal{J}^{\otimes k})_x\right)\hat{} = \varprojlim_Z H^1(E, \mathcal{J}^{\otimes k}{}_{|Z})$$

(cf. the proof of theorem 3.1), so we are left to show $H^1(E, \mathcal{J}^{\otimes k}{}_{|Z}) = 0$ for all positive cycles Z. Again, we can construct a divisor A on Z as a sum of local transverse cuts such that $\deg_Z A = \deg_Z \mathcal{J}^{\otimes k}$.
From the short exact sequence

$$0 \to \mathcal{O}_Z \to \mathcal{O}_Z(A) \to \mathcal{O}_A(A) \to 0$$

we get

$$H^1(E, \mathcal{O}_Z) \to H^1(E, \mathcal{O}_Z(A)) \to H^1(A, \mathcal{O}_A(A)).$$

We have $H^1(E, \mathcal{O}_Z) = 0$ by corollary 3.2 and $H^1(A, \mathcal{O}_A(A)) = 0$ for dimension reasons A.6, hence also $H^1(E, \mathcal{O}_Z(A)) = 0$. ■
This shows

$$H^0(\widetilde{X}', \mathcal{O}_{\widetilde{X}'}(-Z_{\text{num}})) \twoheadrightarrow H^0(E, \mathcal{J}_{|Z_{\text{num}}})$$

and we can fetch a preimage $s' \in H^0(\widetilde{X}', \mathcal{O}_{\widetilde{X}'}(-Z_{\text{num}}))$ of s. By construction, $s'_p \neq 0$, thus $s'_q \neq 0$ for all $q \in U$ for some neighbourhood U of p. Or put differently,

$$(s')_{|U} = Z_{\text{num}|U}.$$

We have already seen $\pi_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$; therefore s' gives rise to a section in $\Gamma(\pi(U), \mathcal{O}_X)$, and thus in $\Gamma(\pi(U), \mathfrak{m}_x)$, since $Z_{\text{num}} \geq E$. □

Lemma 3.9 ([27] 4.18) *The ring* $\bigoplus_{k\geq 0} H^0(E, \mathcal{J}^{\otimes k})$ *is generated in degree* 1, *where* $\mathcal{J} := \mathcal{O}_{Z_{\text{num}}}(-Z_{\text{num}})$.

Proof: We can use local transverse cuts to construct a divisor A on Z_{num} with $\mathcal{O}_{Z_{\text{num}}}(A) \cong \mathcal{J}$. Therefore there exists a global section $s_0 \in H^0(E, \mathcal{J})$ whose divisor of zeros is precisely A. Since we had a lot of freedom in choosing A, we see that the linear system $|A| = |\mathcal{J}|$ is basepoint-free. Thus we can choose a $s \in H^0(E, \mathcal{J})$ such that s provides a local base at every point $q \in A$. We can use s^{k-1} to identify $\mathcal{J}^{\otimes k-1} \otimes \mathcal{O}_A \cong \mathcal{O}_A$. The short exact sequence

$$0 \to \mathcal{O}_{Z_{\text{num}}} \to \mathcal{J} \to \mathcal{O}_A \to 0$$

yields (corollary 3.2)

$$0 \to H^0(E, \mathcal{O}_{Z_{\text{num}}}) \to H^0(E, \mathcal{J}) \to H^0(A, \mathcal{O}_A) \to 0 \tag{7}$$

and (A.6)

$$H^1(E, \mathcal{J}) = 0.$$

Let $s_1, \ldots, s_d \in H^0(E, \mathcal{J})$ map to a basis of $H^0(A, \mathcal{O}_A)$, $d := h^0(A, \mathcal{O}_A)$. Our lemma will follow from the following claim:

$$H^1(E, \mathcal{J}^{\otimes k}) = 0,$$
$$H^0(E, \mathcal{J}^{\otimes k}) = \mathrm{span}_{\mathbb{C}}\{{s_0}^k, {s_0}^{k-l} s^{l-1} s_i : 1 \le l \le k, 1 \le i \le d\}$$

Note that the sections ${s_0}^{k-l} s^{l-1} s_i, l = 1, \dots, k, i = 1, \dots, d$ and ${s_0}^k$ are linearly independent over $\mathbb{C}$.
For $k = 1$ our claim follows from (7) and $h^0(E, \mathcal{O}_{Z_{\mathrm{num}}}) = 1$:

$$h^0(E, \mathcal{J}) = h^0(E, \mathcal{O}_{Z_{\mathrm{num}}}) + h^0(A, \mathcal{O}_A) = 1 + d.$$

For the induction step we argue similarly using

$$\begin{array}{ccccccc} 0 & \to & H^0(E, \mathcal{J}^{\otimes k-1}) & \to & H^0(E, \mathcal{J}^{\otimes k}) & \to & H^0(A, \mathcal{O}_A) \\ & \to & 0 & \to & H^1(E, \mathcal{J}^{\otimes k}) & \to & 0 \end{array}$$

to get

$$h^0(E, \mathcal{J}^{\otimes k}) = h^0(E, \mathcal{J}^{\otimes k-1}) + h^0(A, \mathcal{O}_A) = (k-1)d + 1 + d$$

and

$$H^1(E, \mathcal{J}^{\otimes k}) = 0.$$

□

Proof of the theorem ([27] 4.18): Let $X' = \mathrm{Spec}\, R$ be an affine neighbourhood of $x \in X$ and set $\widetilde{X}' := \widetilde{X} \times_X X'$. By lemma 3.8, we know

$$H^0(\widetilde{X}', \mathfrak{m}_x \cdot \mathcal{O}_{\widetilde{X}}) = H^0(\widetilde{X}', \mathcal{O}_{\widetilde{X}}(-Z_{\mathrm{num}})). \tag{8}$$

We want to generalize (8) to

$$H^0(\widetilde{X}', {\mathfrak{m}_x}^k \cdot \mathcal{O}_{\widetilde{X}}) = H^0(\widetilde{X}', \mathcal{O}_{\widetilde{X}}(-kZ_{\mathrm{num}})). \tag{9}$$

With (9) the proof of our assertion is straightforward: Since

$$H^1(\widetilde{X}', \mathcal{O}_{\widetilde{X}}(-(k+1)Z_{\mathrm{num}})) = 0$$

by corollary 3.2, we have

$$\frac{H^0(\widetilde{X}', {\mathfrak{m}_x}^k \cdot \mathcal{O}_{\widetilde{X}})}{H^0(\widetilde{X}', {\mathfrak{m}_x}^{k+1} \cdot \mathcal{O}_{\widetilde{X}})} = \frac{H^0(\widetilde{X}', \mathcal{O}_{\widetilde{X}}(-kZ_{\mathrm{num}}))}{H^0(\widetilde{X}', \mathcal{O}_{\widetilde{X}}(-(k+1)Z_{\mathrm{num}}))} = H^0(E, \mathcal{O}_{Z_{\mathrm{num}}}(-kZ_{\mathrm{num}})).$$

The Riemann-Roch theorem for curves tells us

$$\begin{array}{rcl} h^0(E, \mathcal{O}_{Z_{\mathrm{num}}}(-kZ_{\mathrm{num}})) & = & 1 - p_a(Z_{\mathrm{num}}) + \deg\, \mathcal{O}_{Z_{\mathrm{num}}}(-kZ_{\mathrm{num}}) \\ & = & 1 - k(Z_{\mathrm{num}})^2, \end{array}$$

that is the leading coefficient of the Hilbert-Samuel polynomial of $(\mathcal{O}_{X',x}, m_x)$ is $-(Z_{\mathrm{num}})^2$.
We will prove (9) by induction, so assume (9) holds for $k < l$. Clearly (8) implies the inclusion

$$H^0(\widetilde{X}', \mathfrak{m}_x{}^l \cdot \mathcal{O}_{\widetilde{X}}) \subseteq H^0(\widetilde{X}', \mathcal{O}_{\widetilde{X}}(-lZ_{\text{num}})).$$

We want to show surjectivity. We take a $g \in H^0(\widetilde{X}', \mathcal{O}_{\widetilde{X}}(-lZ_{\text{num}}))$ and restrict it to $\bar{g} \in H^0(E, \mathcal{O}_{Z_{\text{num}}}(-lZ_{\text{num}}))$. By lemma 3.9 we have a surjection

$$H^0(E, \mathcal{O}_{Z_{\text{num}}}(-Z_{\text{num}})) \otimes H^0(E, \mathcal{O}_{Z_{\text{num}}}(-(l-1)Z_{\text{num}})) \twoheadrightarrow H^0(E, \mathcal{O}_{Z_{\text{num}}}(-lZ_{\text{num}})),$$

i.e. we can write $\bar{g}$ in the form $\bar{g} = \sum_{j=1}^{m} \bar{x}_j \bar{y}_j$ with

$$\bar{x}_j \in H^0(E, \mathcal{O}_{Z_{\text{num}}}(-Z_{\text{num}})) \text{ and}$$
$$\bar{y}_j \in H^0(E, \mathcal{O}_{Z_{\text{num}}}(-(l-1)Z_{\text{num}})) \text{ for } j = 1, \dots, m.$$

Lifting $\bar{x}_j$ and $\bar{y}_j$ to sections on $\widetilde{X}'$

$$x_j \in H^0(\widetilde{X}', \mathcal{O}_{\widetilde{X}}(-Z_{\text{num}})) \text{ and}$$
$$y_j \in H^0(\widetilde{X}', \mathcal{O}_{\widetilde{X}}(-(l-1)Z_{\text{num}})) \text{ for } j = 1, \dots, m$$

gives for $f_2 := \sum_{j=1}^{m} x_j y_j$ by induction hypothesis

$$f_2 \in H^0(\widetilde{X}', \mathfrak{m}_x{}^l \cdot \mathcal{O}_{\widetilde{X}}) \text{ and}$$
$$g - f_2 \in H^0(\widetilde{X}', \mathcal{O}_{\widetilde{X}}(-(l+1)Z_{\text{num}})).$$

Continuing in this fashion gives

$$f_2, \dots, f_p \in H^0(\widetilde{X}', \mathfrak{m}_x{}^l \cdot \mathcal{O}_{\widetilde{X}}) \text{ and}$$
$$g - f_2 - \cdots - f_p \in H^0(\widetilde{X}', \mathcal{O}_{\widetilde{X}}(-(l+p-1)Z_{\text{num}})).$$

Hence it suffices to prove

$$H^0(\widetilde{X}', \mathcal{O}_{\widetilde{X}}(-pZ_{\text{num}})) \subseteq H^0(\widetilde{X}', \mathfrak{m}_x{}^l \cdot \mathcal{O}_{\widetilde{X}}) \text{ for } p \gg 0.$$

The point is that

$$\bigoplus_{k \geq 0} H^0(\widetilde{X}', \mathcal{O}_{\widetilde{X}}(-kZ_{\text{num}}))$$

is a finitely generated R-algebra. Assuming this, let M be the maximal degree in a fixed set of generators. For $p > lM$, each element of $H^0(\widetilde{X}', \mathcal{O}_{\widetilde{X}}(-pZ_{\text{num}}))$ is a sum of products of at least l generators, thus

$$H^0(\widetilde{X}', \mathcal{O}_{\widetilde{X}}(-pZ_{\text{num}})) \subseteq H^0(\widetilde{X}', \mathfrak{m}_x{}^l \cdot \mathcal{O}_{\widetilde{X}}) \text{ for } p > lM.$$

For the proof of the assumption, note that the complete linear system $|\mathcal{O}_{\widetilde{X}'}(-Z_{\text{num}})|$ is free: By the useful fact, $H^1(\widetilde{X}', \mathcal{O}_{\widetilde{X}'}(-2Z_{\text{num}})) = 0$, i.e. we have a surjection

$$H^0(\widetilde{X}', \mathcal{O}_{\widetilde{X}'}(-Z_{\text{num}})) \twoheadrightarrow H^0(E, \mathcal{O}_{Z_{\text{num}}}(-Z_{\text{num}})).$$

We have seen in the proof of lemma 3.8 that $|\mathcal{O}_{Z_{\text{num}}}(-Z_{\text{num}})|$ is free and hence so is $|\mathcal{O}_{\widetilde{X}'}(-Z_{\text{num}})|$. Thus we have a well-defined morphism

$$\phi_{|-Z_{\text{num}}|} : \widetilde{X}' \to X' \times \mathbb{P}^N = \mathbb{P}^N_{X'}.$$

We denote its image by $Y := \text{im } \phi_{|-Z_{\text{num}}|}$. We want to show that Y is closed. Since $\widetilde{X} \to X$ is proper, so is $\widetilde{X}' \to X'$ ([17] II.4.8.(c)). Because of the separatedness of $\mathbb{P}^N_{X'} \to X'$ ([17] II.4.9), we see that $\phi_{|-Z_{\text{num}}|} : \widetilde{X}' \to \mathbb{P}^N_{X'}$ is proper ([17] II.4.8.(e)), in particular $\phi_{|-Z_{\text{num}}|}$ is closed. Hence Y is closed.
The pullback under $\phi_{|-Z_{\text{num}}|}$ of the relatively ample line bundle $\mathcal{O}(1) := \mathcal{O}_{X'} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^N}(1)$ is by definition of $\phi_{|-Z_{\text{num}}|}$ simply $\mathcal{O}_{\widetilde{X}'}(-Z_{\text{num}})$ ([27] 4.18).
Thus it suffices to show that

$$S'(Y) := \bigoplus_{k \geq 0} H^0(Y, \mathcal{O}_Y(k))$$

is finitely generated as a R-algebra. The homogenous coordinate ring $S(Y) = A[x_0, \ldots, x_N]/I(Y)$ of Y is certainly a finitely generated R-algebra. By [17] Ex. II.5.9, there exists a natural graded morphism

$$S(Y) \to S'(Y),$$

which is an isomorphism in high degrees, i.e.

$$S(Y)^d \xrightarrow{\sim} S'(Y)^d \text{ for } d \gg 0,$$

and we are done. □

Corollary 3.10 *For a rational double point (X, x), we have $(Z_{\text{num}})^2 = -2$.*

4 The geometry of the exceptional set E of a resolution of a rational double point

Once the hard work has been done in proving theorems 3.1 and 3.7, it is now easy to say explicitly what configurations can arise for E, if (X, x) is a rational double point.
From now on, we will assume that $\pi : \widetilde{X} \to X$ is a good resolution of a rational double point (X, x) and E its exceptional set.
By proposition 2.3, we have $E_i^2 \leq -1$ for $i = 1, \ldots, n$. If $E_{i_0}^2 = -1$ for some i_0, then $E_{i_0} \cong \mathbb{P}^1$ can be contracted by Castelnuovo's criterion A.4 to give a resolution $\pi' : \widetilde{X}' \to X$ with fewer E_i. (In general, the resolution $\pi' : \widetilde{X}' \to X$ needs not to be good anymore, since the condition 2 in the definition of a good resolution might be violated. However, it is a simple consequence of the following theorem 4.1 and, again, lemma 2.3, that this cannot happen in our case. Note that we do not use this condition 2 in the proof of 4.1.)
Therefore, we can assume $E_i^2 \leq -2$ for $i = 1, \ldots, n$ without loss of generality.

Theorem 4.1 ([9]) *The E_i have self-intersection-number -2.*

Proof: Let K be a canonical divisor on $\widetilde{X}$ ([17] V.1.4.4). The adjunction formula A.2 tells us

$$-E_i \cdot K = E_i^2 + 2 \tag{10}$$

and thus

$$E_i \cdot K \geq 0.$$

We apply the adjunction formula A.2 for Z_{num}

$$2p_a(Z_{\text{num}}) - 2 = (Z_{\text{num}})^2 + Z_{\text{num}} \cdot K,$$

and get by the corollaries 3.3 and 3.10

$$0 = Z_{\text{num}} \cdot K = \sum_{i=1}^{n} r_i (E_i \cdot K) \geq 0, \text{ i.e. } E_i \cdot K = 0.$$

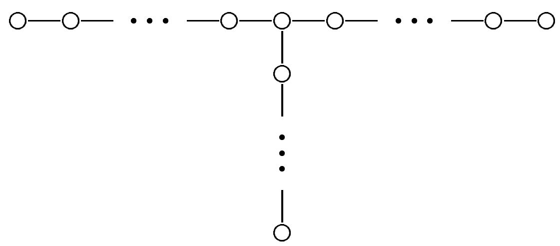
Using (10) again, we see $E_i^2 = -2$. □

We define the *Dynkin diagram* of the resolution $\pi : \widetilde{X} \to X$ to be the weighted dual graph Γ associated to E: The vertices e_i of Γ correspond to the E_i. Whenever E_i and E_j intersect for $i \neq j$, the corresponding vertices are joined by an edge. Finally, we associate to every vertex e_i of Γ the self-intersection-number E_i^2.
Every weighted graph Γ defines a bilinear form $\langle \cdot, \cdot \rangle$ on the free module with the vertices $e_i, i = 1, \ldots, n$ of Γ as basis in the following way: We take

$$\langle e_i, e_i \rangle := \text{the weight of } e_i \text{ and}$$
$$\langle e_i, e_j \rangle := \text{number of edges joining } e_i \text{ and } e_j.$$

The bilinear form of the Dynkin diagram of a resolution is obviously given by the matrix $(E_i \cdot E_j)_{i,j=1,\ldots,n}$ and hence negative definite by proposition 2.3. This puts very strong restrictions on the possible Dynkin diagrams Γ.

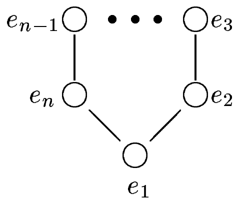
Proposition 4.2 ([9]) *Let Γ be a connected graph weighted by -2 whose associated bilinear form is negative definite. Then Γ is a T-tree $T_{p,q,r}$*



with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.

Proof: Every connected subgraph Γ' of Γ satisfies the hypothesis as well, hence can be

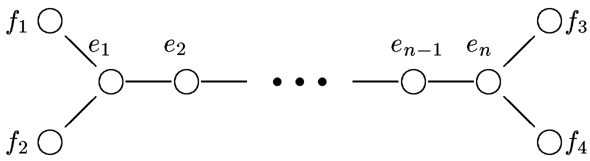
- neither a loop



,

since then $(e_1 + \cdots + e_n)^2 = 0$ contradicting the negative definiteness condition

- nor of the form



,

since then $(2e_1 + \cdots + 2e_n + f_1 + \cdots + f_4)^2 = 0$.

Thus Γ must be of the form $T_{p,q,r}$.
The condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ follows by an elementary argument: With respect to the standard basis given by the vertices of Γ, the associated bilinear form is expressed by the matrix

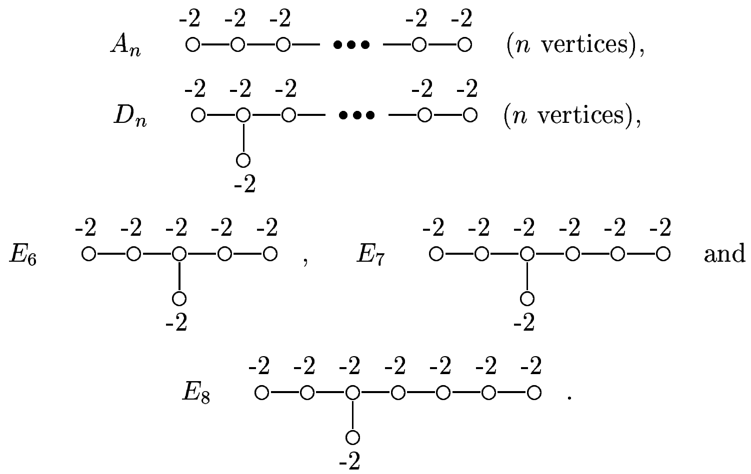
$$\begin{pmatrix}
-2 & 1 & & & & & & & & & \\
1 & -2 & 1 & & & & & & & & \\
 & 1 & -2 & & & & \underset{p,p+q}{\boxed{1}} & & & & \\
 & & & \ddots & & & & & & & \\
 & & & & -2 & 1 & & & & & \\
 & & & & 1 & -2 & \underset{p+q-1,p+q}{\boxed{0}} & & & & \\
 & & & \underset{p+q,p}{\boxed{1}} & & \underset{p+q,p+q-1}{\boxed{0}} & -2 & 1 & & & \\
 & & & & & & 1 & -2 & 1 & & \\
 & & & & & & & 1 & -2 & & \\
 & & & & & & & & & \ddots & 1 \\
 & & & & & & & & & 1 & -2
\end{pmatrix}.$$

But, up to congruence, this is equal to

$$\begin{pmatrix} -\frac{2}{1} &&&&&&&&& \\ & -\frac{3}{2} &&&&&&&& \\ && \ddots &&&&&&& \\ &&& -\frac{p+1}{p} & \underset{p,p+1}{\boxed{1}} &&& \underset{p,p+q}{\boxed{1}} && \\ &&& \underset{p+1,p}{\boxed{1}} & -\frac{q}{q-1} &&&&& \\ &&&&& \ddots &&&& \\ &&&&&& -\frac{2}{1} &&& \\ &&& \underset{p+q,p}{\boxed{1}} &&&& -\frac{r}{r-1} && \\ &&&&&&&& \ddots & \\ &&&&&&&&& -\frac{2}{1} \end{pmatrix}.$$

Now, this matrix is congruent to a diagonal matrix with negative main diagonal entries, except maybe a single one $1 - p^{-1} - q^{-1} - r^{-1}$. □

Corollary 4.3 *The Dynkin diagram associated to a rational double point* (X, x) *must be one of the following diagrams*



We will see in section 5.2 that all Dynkin diagrams actually occur. We say that a rational double point is of type A_n, D_n or E_n according to its Dynkin diagram.

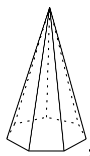
5 Example: The singularities $\mathbb{C}/G$ for finite $G \subset \mathbf{SL}(2, \mathbb{C})$

After a bit of the theory of rational double points has been presented, we want to study an example.

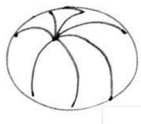
5.1 Conjugacy classes of finite subgroups of $\mathbf{SL}(2, \mathbb{C})$

As a preliminary, we recall briefly the classification of conjugacy classes of finite subgroups of $\mathrm{SL}(2, \mathbb{C})$. We consider first $\mathrm{SO}(3, \mathbb{R})$. Up to conjugacy, the finite subgroups of $\mathrm{SO}(3, \mathbb{R})$ are the rotational symmetry groups of

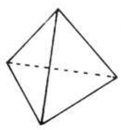
- a pyramid (giving the *cyclic subgroups* C_n)

  ,

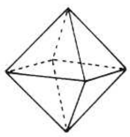
- an orange (corresponding to the *dihedral subgroups* D_n)

  ,

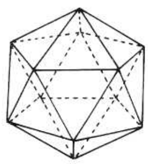
- and the Platonic solids, which give
  - the *tetrahedral subgroup* $T = A_4$

    ,

  - the *octahedral subgroup* $O = S_4$

    ,

  - and the *icosahedral subgroup* $I = A_5$

    ,

respectively [30].
If we identify $S^2 \cong \mathbb{P}^1$, we get an inclusion of the group of isometries of $\mathbb{P}^1$ (with respect to the usual metric) into the group of conformal transformations

$$\mathrm{SO}(3, \mathbb{R}) \subset \mathbb{P}\mathrm{GL}(2, \mathbb{C}).$$

Under the double cover

$$\rho : \mathrm{SL}(2,\mathbb{C}) \twoheadrightarrow \mathbb{P}\mathrm{GL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm 1\}$$

this inclusion corresponds to

$$\rho^{-1}(\mathrm{SO}(3,\mathbb{R})) = \mathrm{SU}(2,\mathbb{C}).$$

Since for any finite subgroup G of SL(2,$\mathbb{C}$) we can find a G-invariant Hermitian metric by averaging an arbitrary one, every finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$ is conjugated to a subgroup of $\mathrm{SU}(2,\mathbb{C})$. Hence it corresponds to a finite subgroup of $\mathrm{SO}(3,\mathbb{R})$, unless it is a cyclic group of odd order. Thus we have derived the following classification of the conjugacy classes of finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ ([20] II §1):

- the cyclic subgroup of order n C_n,
- the binary dihedral subgroups $\widetilde{D}_n = \rho^{-1}(D_n)$,
- the binary tetrahedral, octahedral and icosahedral subgroup $\widetilde{T} = \rho^{-1}(T)$, $\widetilde{O} = \rho^{-1}(O)$ and $\widetilde{I} = \rho^{-1}(I)$ respectively.

5.2 The singularities $\mathbb{C}^2/G$

Now let G be any of these subgroups; the affine orbit variety

$$\mathbb{C}^2/G = \mathrm{Spec}\ \mathbb{C}[x_1, x_2]^G$$

has an isolated singularity at the origin. The singularities obtained in this fashion are all rational double points [9]. It is a result from classical invariant theory that these singularities embed in codimension one

$$\mathbb{C}^2/G = \mathrm{Spec}\ \mathbb{C}[x,y,z]/(f) \text{ where } f \in \mathbb{C}[x,y,z].$$

See for example Klein's influential book [19], or also [11] 5.39. For a modern treatment, we refer to [8], p. 5. The following table 1 contains the basic information about these singularities.

5.3 The icosahedral case $G = \widetilde{I}$

We will sketch the proof of the assertions made so far in the special case $G = \widetilde{I}$. We see that

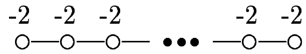
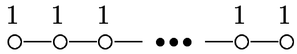
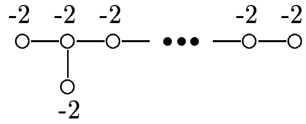
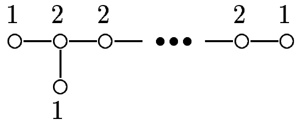
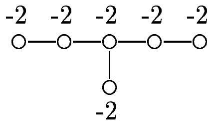
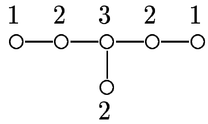
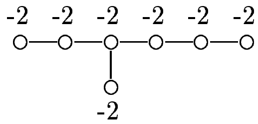
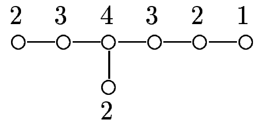
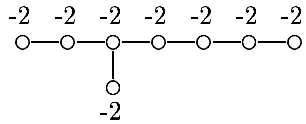
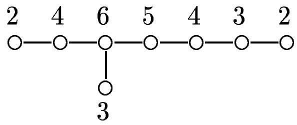
$$X := \mathbb{C}^2/\widetilde{I} = \mathrm{Spec}\ \mathbb{C}[x,y,z]/(x^2 + y^3 + z^5)$$

has a singularity at $x_0 := (0,0,0)$, which must be a double point, since

$$\mathrm{rank}_{\mathbb{C}} \frac{(x,y,z)^k + (x^2+y^3+z^5)}{(x,y,z)^{k+1} + (x^2+y^3+z^5)} \leq 2k+1.$$

We want to show that (X, x_0) is a **rational** double point. Indeed, we will show how to resolve the singularity (X, x_0) ([20] IV §9, [8] p. 15). Blowing up $\mathbb{C}^2$ instead of $\mathbb{C}^2/\widetilde{I}$ will reveal the relevant information more clearly. The blow-up of $\mathbb{C}^2$ at the origin is known to be $\mathbf{v}(\mathcal{O}_{\mathbb{P}^1}(-1))$ ([17] V.3.1); here $\mathbf{v}(\mathcal{F})$ denotes

Table 1: The singularities $\mathbb{C}^2/G = \text{Spec } \mathbb{C}[x,y,z]/(f)$ for finite $G \subset \text{SL}(2,\mathbb{C})$

G	f	Name	Dynkin diagram	Z_{num} [a]
C_n	$x^n + y^2 + z^2$	A_{n-1}	 ($n-1$ vertices)	
$\widetilde{D}_n$	$x^{n+1} + xy^2 + z^2$	D_{n+2}	 ($n+2$ vertices)	
$\widetilde{T}$	$x^2 + y^3 + z^4$	E_6		
$\widetilde{O}$	$x^2 + y^3 + yz^3$	E_7		
$\widetilde{I}$	$x^2 + y^3 + z^5$	E_8		

[a] The number above a vertex denotes the multiplicity of the corresponding projective line $E_i \cong \mathbb{P}^1$ in the numerical cycle Z_{num}.

the vector bundle determined by the locally trivial sheaf $\mathcal{F}$. The quotient of $\mathbf{v}(\mathcal{O}_{\mathbb{P}^1}(-1))$ by $\{\pm 1\}$ is $\mathbf{v}(\mathcal{O}_{\mathbb{P}^1}(-2))$:

$$\begin{gathered} \mathbf{v}(\mathcal{O}_{\mathbb{P}^1}(-1))/\{\pm 1\} = \{(w_1 : w_2; z_1, z_2) \in \mathbb{P}^1 \times \mathbb{C}^2 : w_1 z_2 = w_2 z_1\}/\{\pm 1\} \cong \\ \{(w_1 : w_2; z_1, z_2) \in \mathbb{P}^1 \times \mathbb{C}^2 : w_1^2 z_2 = w_2^2 z_1\} = \mathbf{v}(\mathcal{O}_{\mathbb{P}^1}(-2)) \\ (w_1 : w_2; z_1, z_2) \mapsto (w_1 : w_2; z_1^2, z_2^2). \end{gathered}$$

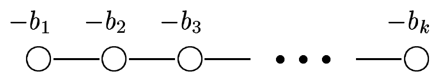
Note that the $\widetilde{I}$-action on $\mathbb{C}^2$ lifts to an action on $\mathbf{v}(\mathcal{O}_{\mathbb{P}^1}(-1))$. Thus we have the following commutative diagram

$$\begin{array}{ccccc} \mathbf{v}(\mathcal{O}_{\mathbb{P}^1}(-2)) \cong \mathbf{v}(\mathcal{O}_{\mathbb{P}^1}(-1))/\{\pm 1\} & \longleftarrow & \mathbf{v}(\mathcal{O}_{\mathbb{P}^1}(-1)) & \xrightarrow{\sigma} & \mathbb{C}^2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{v}(\mathcal{O}_{\mathbb{P}^1}(-2))/I & \xleftarrow{\sim} & \mathbf{v}(\mathcal{O}_{\mathbb{P}^1}(-1))/\widetilde{I} & \xrightarrow{\bar{\sigma}} & \mathbb{C}^2/\widetilde{I} \end{array}.$$

In particular, $\bar{\sigma}^{-1}(x_0)$ is a copy of $\mathbb{P}^1$ with self-intersection-number -2. A precise analysis of the I-action on $\mathbf{v}(\mathcal{O}_{\mathbb{P}^1}(-2))$ shows, that I acts on the zero-section $S^2 \cong \mathbb{P}^1 \subseteq \mathbf{v}(\mathcal{O}_{\mathbb{P}^1}(-2))$ in the usual way as rotations which leave an inscribed icosahedron invariant. Furthermore the action of I on $T^*\mathbb{P}^1 \cong \mathbf{v}(\mathcal{O}_{\mathbb{P}^1}(-2))$ is simply the cotangent action induced by the action of I on $\mathbb{P}^1$ ([20] IV §7). Now, the group I is acting free on S^2, except on three exceptional orbits, which consist of the vertices, the mid-edge-points and the mid-face-points of the inscribed icosahedron, respectively. Moreover, we see that these three orbits are also the only exceptional orbits for the action of I on $\mathbf{v}(\mathcal{O}_{\mathbb{P}^1}(-2))$. Therefore, the quotient variety $\mathbf{v}(\mathcal{O}_{\mathbb{P}^1}(-2))/I$ is smooth except at the three points corresponding to these orbits. An explicit calculation using local coordinates shows that these three singular points are cyclic quotient singularities of type $(5,4)$, $(3,2)$ and $(2,1)$, respectively ([8] p. 17), this notion being defined as follows: A *cyclic quotient singularity of type* (n,q) is a singularity, which is isomorphic to $\mathbb{C}^2/\mu_{n,q}$ where $\mu_{n,q}$ is the cyclic group generated by $\begin{pmatrix} \xi & 0 \\ 0 & \xi^q \end{pmatrix}$ for a n-th root of unity ξ. Note that the numbers 5, 3, 2 are the ramification indices at the ramification points of the map

$$S^2 \twoheadrightarrow S^2/I$$

corresponding to the exceptional orbits.
A cyclic quotient singularity of type (n,q) can be resolved by the Hirzebruch-Jung algorithm using successive blow-ups of points ([12] 2.6). The exceptional set of a resolution of a cyclic quotient singularity obtained in this way is a bunch of rational curves; the associated Dynkin diagram is of the form

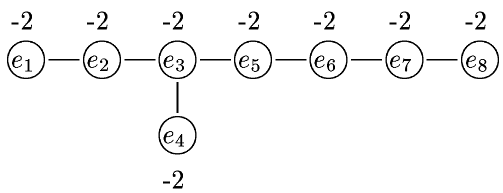


,

where the b_i's are calculated by a modified Euclidean algorithm

$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \frac{1}{b_k}}}.$$

Applying the Hirzebruch-Jung algorithm three times for the three singular points we got, we obtain a resolution $\pi : \widetilde{X} \to X$ with associated Dynkin diagram



The numerical divisor Z_{num} is easily verified to be

$$Z_{\text{num}} = 2E_1 + 4E_2 + 6E_3 + 3E_4 + 5E_5 + 4E_6 + +3E_7 + 2E_8.$$

A straightforward computation using equation (6) shows

$$p_a(Z_{\text{num}}) = 0,$$

i.e. the singularity (X, x_0) is rational by the criterion of theorem 3.4.

6 Changing to the complex analytic category

The examples introduced in the preceding section already exhaust all possibilities of rational double points up to isomorphism in the complex analytic category. Of course, such a statement cannot be true in the category of algebraic varieties, since rational double points can live on the various kinds of surfaces and the birationality class of the surface is encoded locally due to the coarseness of the Zariski topology.
We give the following (simplified) definition of the *complex analytic category*:

- Its objects are called *complex analytic spaces* and can be constructed as follows. Let U be a simply connected open subset of $\mathbb{C}^n$, $\mathcal{O}_U$ the sheaf of complex analytic functions on U, and $\mathcal{J}_X$ a sheaf of ideals on $(U, \mathcal{O}_U)$. Denote by $X \subseteq U$ the zeroset of $\mathcal{J}_X$ equipped with the **standard topology** and set $\mathcal{O}_X := \mathcal{O}_U/\mathcal{J}_X$. The pair $(X, \mathcal{O}_X)$ is then a complex analytic space. (For the general definition one allows such simple building blocks to be glued together as in the definition of schemes.)
- a *morphism of complex analytic spaces* $f : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ is a continuous map $f : X \to X'$ such that $f^* : \mathcal{O}_{X'} \to \mathcal{O}_X$ is well-defined.

To a great extent, the complex analytic category is similar to the category of algebraic varieties: For example, stalks $\mathcal{O}_{X,x}$ are noetherian local rings and for reduced complex analytic spaces $(X, \mathcal{O}_X)$ Rückert's Nullstellensatz holds ([20] III §8).

7 Tautness of rational double points

7.1 Definition and theorem

To pick up the question of classifying rational double points in the complex analytic category, we introduce the notion of tautness.
Let (X, x) be a two-dimensional normal singularity with a good resolution, whose exceptional set is a bunch of rational curves $\mathbb{P}^1$ and let Γ be its Dynkin diagram.
We say (X, x) is *taut*, if up to analytic isomorphism, (X, x) is the unique such singularity, that has a good resolution with a bunch of rational curves $\mathbb{P}^1$ as exceptional set and Γ as its Dynkin diagram [4].
We have the following theorem.

Theorem 7.1 *The singularities listed in table 1 are taut.*

This gives us a complete classification of rational double points up to analytic isomorphism. There are several proofs available for theorem 7.1.

7.2 Tjurina's proof

Maybe the most natural one is the proof by Tjurina [31]: Suppose there were another singularity (X', x') with an exceptional set consisting only of rational curves and the same Dynkin diagram, i.e. with an isomorphic exceptional variety $(E', \mathcal{O}_{E'}) \cong (E, \mathcal{O}_E)$. Then a sufficient condition for the existence of an isomorphism of neighbourhoods of E and E' is by [15] Thm. 3, that $(E, \mathcal{O}_{nE})$ and $(E', \mathcal{O}_{nE'})$ are isomorphic for n large enough. The proof given in [31] proceeds by induction. Assuming we are given some isomorphism of $(E, \mathcal{O}_Z)$ and $(E', \mathcal{O}_{Z'})$ (here Z and Z' are exceptional divisors supported on $E = \bigcup_{i=1}^n E_i$, $E' = \bigcup_{i=1}^n E'_i$, respectively), then this can be extended to an isomorphism of $(E, \mathcal{O}_{Z+E_i})$ and $(E', \mathcal{O}_{Z'+E'_i})$, unless some obstruction occurs, which lies in some cohomology group [15]. Grauert argues, that if all these cohomology groups vanish, then the singularity in question is taut. In general, these cohomology groups do not vanish and Tjurina's proof is more subtle. Essentially, he shows that the cohomology groups are too small to put obstructions on the lifting of every possible isomorphism of $(E, \mathcal{O}_Z)$ and $(E', \mathcal{O}_{Z'})$.

7.3 Brieskorn's first proof

For the sake of historical correctness, we mention that the first proof of theorem 7.1 was given by Brieskorn ([3] Satz 1). He showed that rational double points can be resolved by blowing up points alone, that is, it is not necessary to normalize or blow up curves. Such singularities are called *absolutely isolated* and were studied by Kirby ([18] 2.6, 2.7), who gave a classification of absolutely isolated double points: they are precisely those listed in table 1.

7.4 Brieskorn's second proof

However, we want to sketch another proof of theorem 7.1, which, also due to Brieskorn [4], is of compelling beauty and combines ideas from different fields of mathematics:
The *local fundamental group* of (X, x) is defined as

$$\pi_{X,x} := \varprojlim_{U} \pi_1(U \setminus \{x\})$$

where U runs over all neighbourhoods of $x \in X$ ([4] §2). Equivalently, we can calculate $\pi_{X,x}$ as

$$\pi_{X,x} \cong \varprojlim_{\widetilde{U}} \pi_1(\widetilde{U} \setminus \{x\})$$

where the limit is now taken over all neighbourhoods $\widetilde{U}$ of $E \subset \widetilde{X}$. To actually compute $\pi_{X,x}$ it is sufficient to work out π_1 for a *good* neighbourhood U. According to [26], a neighbourhood U of $x \in X$ is called *good*, if there exists a neighbourhood basis $\{U_i\}$ for x such that $U_i \setminus \{x\}$ is a deformation retract of $U \setminus \{x\}$ for all i. Such a good neighbourhood has the homotopy type of a tubular neighbourhood M of E in the sense of Mumford [24]. Intuitively spoken, a tubular neighbourhood is a levelset of the potential distribution due to a uniform charge on E. Mumford studied these tubular neighbourhoods M and showed that they are built out of standard pieces $S^1 \times S^1 \times [0, 1]$ "plumbed" together in a certain fashion determined by $(E_i \cdot E_j)_{i,j=1,\dots,n}$. This description and the Seifert-van-Kampen theorem enables him to give a presentation for $\pi_1(M)$ in terms of generators and relations. (The ideas of his proof can also be found in [20] IV §§10 - 14 on plumbed surfaces.) It turns out that for the intersection matrices $(E_i \cdot E_j)_{i,j=1,\dots,n}$ of the resolutions of rational double points this group $\pi_{X,x} = \pi_1(M)$ is finite.

A rational double point has finite local fundamental group.

For the following see [4] Satz 2.8, [26] Thm. 3. From a merely topological point of view, $x \in X$ possesses a neighbourhood U with $U' := U \setminus \{x\}$ having a finite universal cover $V' \twoheadrightarrow U'$. This can be uniquely extended to a ramificated cover $V \twoheadrightarrow U$ by adding a point y to V'. Moreover, we can equip V with a normal analytic structure such that $V \twoheadrightarrow U$ becomes an analytically ramificated cover. Since V' is simply connected, we see $\pi_{V,y} = 1$. By another fundamental theorem in Mumford's paper ([24] p. 18), this shows the non-singularity of V at y. Now $\pi_{X,x}$ is operating via cover transformations on V', hence also on V with fixed point y. We need another definition to state our results so far.

Definition and Proposition 7.2 (Two-dimensional quotient singularities) *For a neighbourhood V of the origin O in $\mathbb{C}^2$ and a finite group G of analytic automorphisms of V fixing O, the quotient space V/G has the structure of a normal complex analytic surface and the projection $V \twoheadrightarrow V/G$ is analytic [4]. We say that U is a* two-dimensional quotient singularity, *if U is isomorphic to a singularity of the form V/G.*

We have just seen:

A rational double point is a quotient singularity.

By a simple linearization argument ([4] Lemma 2.2), we can restrict ourselves to the study of quotient singularities of the form $\mathbb{C}^2/G$ where G is a finite subgroup of $\mathrm{GL}(2,\mathbb{C})$. Obviously, conjugated subgroups yield isomorphic quotient spaces. The conjugacy classes of finite subgroups of $\mathrm{GL}(2,\mathbb{C})$ have been listed by Du Val ([11] §21). Later, Prill showed that only a particular class of finite subgroups has to be studied: the so-called *small* subgroups [26]. He also classified them ([26] Satz 2.3). Using Prill's results, Brieskorn gave a complete classification of the quotient spaces that can arise in terms of the Dynkin diagram of their resolution ([4] p. 348). This shows that two-dimensional quotient singularities are taut and finishes Brieskorn's proof.

8 Seven characterizations of rational double points

The following remark allows us to use the intermediate results in the above discussion 7.4 to give alternative characterizations of rational double points in the analytic category.

Remark 8.1 *It can be shown that $\mathbb{C}^2/G$, for $G \subset \mathrm{GL}(2,\mathbb{C})$ finite, embeds in codimension one if and only if G is a subgroup of* $\mathrm{SL}(2,\mathbb{C})$ ([9] cor. 5.3).

Theorem 8.2 ([9])
Let (X, x) be a normal surface singularity that **embeds in codimension one**. *Then the following conditions are equivalent in the complex analytic category.*

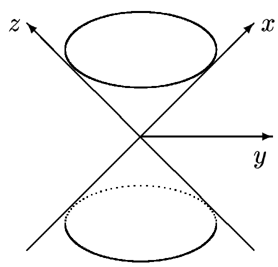
1. *(X, x) is a rational double point.*
2. *(X, x) has a good resolution with an exceptional set consisting of rational curves with self-intersection-number -2.*
3. *(X, x) has a good resolution with an exceptional set consisting of rational curves and a Dynkin diagram listed in table 1.*
4. *The local fundamental group of (X, x) is finite.*
5. *(X, x) is a two-dimensional quotient singularity.*
6. *(X, x) is isomorphic to $\mathbb{C}^2/G$ for finite $G \subset \mathrm{GL}(2, \mathbb{C})$.*
7. *(X, x) is isomorphic to one of the affine varieties studied in section 5.*

Remark on the proof: We have already seen a proof of the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and presented some ideas for (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6). In the example 5.3, we studied a special case of (7) $\Rightarrow$ (1). The last implication (6) $\Rightarrow$ (7) is precisely remark 8.1.

In his survey article [9], Durfee presents further ten characterizations of rational double points. The characterizations he gives provide a connection of rational double points for example with weighted homogeneous polynomials, vanishing cycles, a certain limit involving volumes, monodromy groups and Morse functions. A more number-theoretical characterization in terms of almost factorial rings (fast-faktoriell) rings is due to Brieskorn ([4] Satz 1.5). Finally, a link with elementary catastrophes is discussed in a survey article by Slodowy ([30] 9).

9 Example: The conical double point

Although we did not give a proof of theorem 8.2, we shall at least study an example to illustrate the phenomena encountered there. We will consider the *conical double point* $X := V(f) \subset \mathbb{C}^3$ where $f = xz - y^2 \in \mathbb{C}[x, y, z]$. Note that after a change of variables $xz - y^2$ becomes $x_1^2 + x_2^2 + x_3^2$; so X is just the surface singularity labeled A_1 from table 1. X is a double cone with vertex a rational double point:

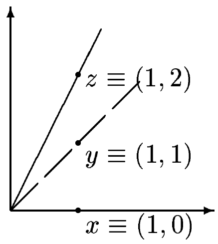


.

Obviously, X has a normal singularity at $x = (0,0,0)$ and X is embedded in codimension one. (X,x) is a double point, since

$$\operatorname{rank}_{\mathbb{C}} \frac{(x,y,z)^k + (f)}{(x,y,z)^{k+1} + (f)} = 2k+1,$$

i.e. the leading coefficient of the Hilbert-Samuel polynomial of the local ring $\mathcal{O}_{X,x}$ at x it two.
Furthermore, (X,x) is absolutely isolated, because the singularity can be resolved by a single blow-up at x, as can be easily seen by the toric description of X [12]:



.

We can work out the blow-up explicitly and obtain

$$\widetilde{X}' \subset \underset{(x,y,z;p:q:r)}{\mathbb{A}^3 \times \mathbb{P}^2}$$

given by equations

$$xz - y^2, pr - q^2, py = qx, pz = rx, qz = ry.$$

$\widetilde{X}'$ is isomorphic to

$$\widetilde{X} \subset \underset{(x,z;u:v)}{\mathbb{A}^2 \times \mathbb{P}^1}$$

cut out by

$$xv^2 = zu^2$$

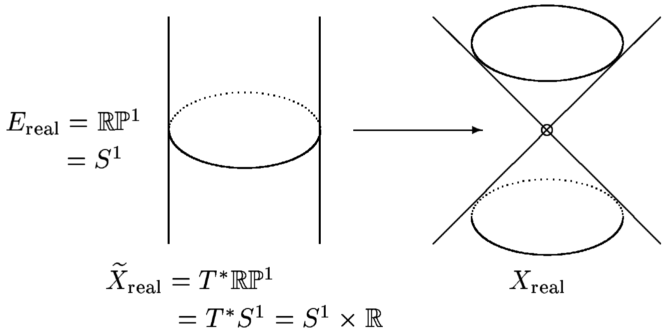
via

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\sim} & \widetilde{X}' \\ (x,z;u:v) & \mapsto & (x, \frac{v}{u}x \text{ or } \frac{u}{v}z, z; u^2 : uv : v^2) \end{array}$$

Now $\widetilde{X}$ is a line bundle on $\mathbb{P}^1$ whose zero-section has self-intersection-number -2. Thus $\widetilde{X}$ is just the line bundle on the projective line associated to the sheaf $\mathcal{O}_{\mathbb{P}^1}(-2)$, i.e. the cotangent bundle $T^*\mathbb{P}^1$. See also [20] IV §7.
Since $\widetilde{X}$ is a smooth variety, we have a resolution of (X,x)

$$\pi : \widetilde{X} \cong \widetilde{X}' \twoheadrightarrow X.$$

The exceptional set E of π is precisely the zero-section of $\widetilde{X}$, hence isomorphic to $\mathbb{P}^1$. Moreover $E^2 = -2$. The real picture reflects the situation very nicely



For the numerical cycle we get $Z_{\text{num}} = E$, i.e. (X, x) is rational by theorem 3.4, and characterization (1) is verified.
As already mentioned in section 5.2, X is isomorphic to the affine orbit variety $\mathbb{C}^2/\{\pm 1\}$, where we write -1 for the reflection in the origin of the complex plane $\mathbb{C}^2$. This corresponds to characterizations (5), (6) and (7).
Finally, let us calculate the local fundamental group $\pi_{X,x}$. We have a covering map

$$\mathbb{C}^2 \setminus \{O\} \twoheadrightarrow X \setminus \{x\}$$

with covering transformation group $\{\pm 1\}$. For every $\{\pm 1\}$-invariant simply connected neighbourhood U of $O \in \mathbb{C}^2$, we observe that $U \setminus \{O\}$ is also simply connected, hence

$$\pi_1(U \setminus \{O\})/\{\pm 1\} = \{\pm 1\}.$$

But such an U can be chosen arbitrarily small, thus

$$\pi_{X,x} = \{\pm 1\},$$

which is finite and shows characterization (4).

10 Lie groups and rational double points

The "A_n-D_n-E_n" - labeling of the various types of rational double points was actually borrowed from the classification theory of Lie groups. In this last section we will sketch some of the deep connections between Lie groups and rational double points. Essentially, we shall give a summary of [30] 10.

10.1 Dynkin diagrams of simple Lie groups

A connected complex Lie group is called *(almost) simple*, if it contains no normal subgroup of positive dimension. In their classification theory, the simply connected simple Lie groups play a special rôle as their universal coverings (which are finite ([30] 10)). These groups are classified by their corresponding Dynkin diagrams [13]. Surprisingly, the Dynkin diagrams of table 1 occur again.
We recall briefly the relevant part of this classification; note that most of the following facts hold in a more general context ([13], [29] 3.1).

Let G be a simply connected simple Lie group of rank r and $\mathfrak{g}$ its Lie algebra. We fix a maximal torus $T \cong (\mathbb{C}^*)^r$ of G with *character group*

$$X^*(T) = \mathrm{Hom}(T, \mathbb{C}^*) = \mathbb{Z}^r.$$

We denote the normalizer of T in G by $N_G(T)$. The group $W := N_G(T)/T$ is finite and is called the *Weyl group* of G with respect to T.
The restriction of the adjoint representation of G on $\mathfrak{g}$ to T has eigenspaces $\mathfrak{g}_\alpha$ on which T acts by the character $\alpha \in X^*(T)$ and we obtain the *Cartan decomposition* of $\mathfrak{g}$

$$\mathfrak{g} = \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}_\alpha.$$

The finite set $\Sigma := \{\alpha \in X^*(T) : \alpha \neq 0, \mathfrak{g}_\alpha \neq \{0\}\}$ is called the *root space.* Clearly, Σ is invariant under the action of W.
We can define a W-invariant scalar product $\langle \cdot, \cdot \rangle$ on $X^*(T)$ (called the *Killing form*) such that the elements of W become reflections in the hyperplane perpendicular to a root α

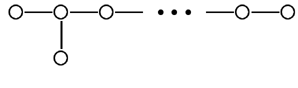
$$\beta \mapsto \beta - \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha.$$

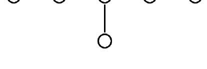
Here $\frac{2\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle}$ must be integer. (The main tool in proving this and similar facts is identifying the subalgebra $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ with $\mathfrak{sl}(2, \mathbb{C})$ and applying the representation theory of $\mathfrak{sl}(2, \mathbb{C})$.)
The geometry of how Σ sits in the Euclidean lattice $(X^*(T), \langle \cdot, \cdot \rangle)$ is very rigid. For example

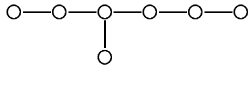
$$4 \cos^2 \angle(\alpha, \beta) = \frac{4\langle \alpha, \beta \rangle^2}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle}$$

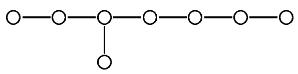
must be an integer between zero and four, i.e. there are just a few possibilities for the angle between two roots α and β.
By choosing a direction $l \in T$ (in general position with respect to Σ) we can specify the *positive roots* $\alpha \in \Sigma$ to be those with $\alpha(l) > 0$. In particular, we can focus on *simple roots*: these are positive roots that are not the sum of two other positive roots. The system of simple roots gives rise to a Dynkin diagram, where we take a vertex for each simple root and join two vertices by exactly $4 \cos^2 \angle(\alpha, \beta)$ lines. If we insist on all roots α having the same length $\langle \alpha, \alpha \rangle$, the only possibilities for the Dynkin diagram are

A_n (n vertices),

D_n  (n vertices),

E_6  ,

E_7  and

E_8  .

These diagrams A_n, D_n, E_6, E_7 and E_8 actually occur for the simply connected simple Lie groups corresponding to the classical Lie algebras $\mathfrak{sl}(n+1,\mathbb{C})$, $\mathfrak{so}(2n,\mathbb{C})$ and the exceptional Lie algebras $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$, respectively.
It can be shown that a simply connected simple Lie group can be recovered from its Dynkin diagram.

10.2 A theorem of Brieskorn

Let G be a simply connected simple Lie group. We consider the quotient H of G by its adjoint action in the category of algebraic varieties $p : G \twoheadrightarrow H$. There is an explicit way to describe p. Let r be the rank of G and $\rho_i : G \to \mathrm{GL}(V_i), i = 1, \ldots, r$ be the r fundamental irreducible representations of G on finite-dimensional vector spaces. Then the character map

$$\begin{aligned} \xi : G &\to \mathbb{C}^r \\ g &\mapsto (\ldots, \mathrm{trace}_{V_i}\rho_i(g), \ldots) \end{aligned}$$

coincides with p.

Example: For $G = \mathrm{SL}(n,\mathbb{C})$ we have rank $G = n-1$ and the $n-1$ fundamental irreducible representations are given by the exterior powers

$$V_i = \bigwedge^i \mathbb{C}^n .$$

The corresponding characters are, up to sign, just the non-trivial coefficients of the characteristic polynomial

$$\mathrm{char}(g) = \det(\lambda - g) = \lambda^n - \mathrm{trace}(g)\lambda^{n-1} + \mathrm{trace}(\wedge^2 g)\lambda^{n-2} - + \ldots .$$

Thus we can regard χ as associating to $g \in \mathrm{SL}(2,\mathbb{C})$ its characteristic polynomial.

The point is now to study the *unipotent variety*

$$\mathrm{Uni}(G) := p^{-1}(p(e)).$$

Example: For $G = \mathrm{SL}(n,\mathbb{C})$ the unipotent variety consists precisely of the unipotent matrices.

The variety $\mathrm{Uni}(G)$ is a finite union of conjugacy classes and contains a unique conjugacy class of dimension $d := \dim G - r$ (since p is flat) — the *regular class*. The complement of the regular class in $\mathrm{Uni}(G)$ is the closure of a unique conjugacy class of dimension $d-2$ — the *subregular class* $\mathrm{Sub}(G)$.
We choose a $S \subseteq G$ such that

$$\begin{aligned} &S \text{ is smooth of dimension } \dim G - d + 2, \\ &S \cap \mathrm{Sub}(G) = \{x\} \text{ and} \\ &T_x S + T_x \mathrm{Sub}(G) = T_x G, \end{aligned}$$

i.e. we require that S is a *slice* of codimension $d-2$ transversal to $\mathrm{Sub}(G)$ at an element $x \in \mathrm{Sub}(G)$.
Let $X := S \cap \mathrm{Uni}(G)$.
The following theorem was conjectured by Grothendieck and proved by Brieskorn.

Theorem 10.1 ([6], [30] 10) *If G is a simply connected simple Lie group of type A_n, D_n or E_n, then (X, x) is a rational double point.*

Example: We want to illustrate this theorem in the simplest possible case $G = \mathrm{SL}(2, \mathbb{C})$. All regular unipotent elements are conjugate to the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and there exists just a single subregular unipotent element

$$x := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

As a transversal slice we can simply take $S := \mathrm{SL}(2, \mathbb{C})$. Now $p = \chi$ is given by the trace

$$\begin{aligned} \chi : \mathrm{SL}(2, \mathbb{C}) &\to \mathbb{C} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto a + d \end{aligned}$$

and we get

$$\begin{aligned} X &= S \cap \mathrm{Uni}(G) \\ &= \mathrm{Uni}(G) = \chi^{-1}\left(\chi\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \left\{\begin{pmatrix} 1+x & y \\ z & 1+u \end{pmatrix} : x + u = 0 \text{ and } xu - yz = 0\right\} \\ &= \{(x, y, z) : x^2 + yz = 0\}, \end{aligned}$$

i.e. X has a conical double point at $x = (0, 0, 0)$.

10.3 Resolutions of rational double points in the Lie group context

A closed subgroup $P \subseteq G$ is called *parabolic*, if the quotient space G/P is a projective variety. The minimal parabolic subgroups are the *Borel subgroups*. All Borel subgroups are conjugate to each other in G and the normalizer $N_G(P)$ of a parabolic subgroup coincides with P. Thus the set of all Borel subgroups $\mathcal{B}$ becomes a projective variety

$$\mathcal{B} = G/B$$

where B is any Borel subgroup of G. More generally, $\mathcal{P} := G/P$ may be identified with the set of subgroups conjugate to the parabolic subgroup P.

Example: The parabolic subgroups of $\mathrm{SL}(n, \mathbb{C})$ are exactly the stabilizer of the flags

$$0 \subset V_{i_1} \subset \cdots \subset V_{i_k} \subset \mathbb{C}^n \text{ with } \mathrm{rank}_{\mathbb{C}} V_{i_j} = i_j, j = 1, \ldots, k.$$

Hence the Borel subgroups correspond to maximal flags

$$0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n,$$

i.e. are conjugate to the subgroup of upper triangular matrices.

It was Springer who showed that the natural projection from the incidence variety

$$I := \{(x, B) \in \mathrm{Uni}(G) \times \mathcal{B} : x \in B\}$$

to $\mathrm{Uni}(G)$ is a G-equivariant resolution of the singularities of $\mathrm{Uni}(G)$

$$\pi : I \twoheadrightarrow \mathrm{Uni}(G).$$

Let G, x, S, X be as in theorem 10.1. It is a consequence of the G-equivariance of π that the restriction

$$\pi : \widetilde{X} := \pi^{-1}(X) \twoheadrightarrow X$$

is again a resolution. In fact, it is a *minimal* one, that is, we cannot apply theorem A.4 to obtain a resolution with smaller exceptional set.
We can interpret the exceptional set

$$E := \pi^{-1}(x)$$

in two different ways.
On one hand, we know from theorem 10.1, that (X, x) is a rational double point. Hence E must be a bunch of projective lines $\mathbb{P}^1$ intersecting each other as prescribed by the Dynkin diagram Γ of (X, x).
On the other hand, we can write

$$E = \{(x, B) \in \{x\} \times \mathcal{B} : x \in B\}.$$

The vertices of the Dynkin diagram Γ_G of G correspond to the simple roots of G (after a maximal torus T_0 and a direction $l \in T_0$, or equivalently, a Borel subgroup $B_0 \supset T_0$ have been specified). Let P_α be the minimal proper (i.e. non-Borel) parabolic subgroup generated by B_0 and the root subgroup $U_{-\alpha}$, where α is a simple root. Because of $N_G(P_\alpha) = P_\alpha$, we can identify the set of subgroups conjugate to P_α with the projective variety

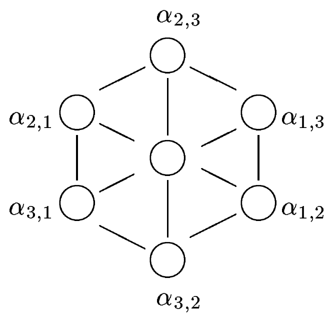
$$\mathcal{P}_\alpha := G/P_\alpha.$$

The natural map

$$f_\alpha : \mathcal{B} \cong G/B_0 \to \mathcal{P}_\alpha \cong G/P_\alpha$$

maps each Borel subgroup B to the unique parabolic subgroup $P \in \mathcal{P}_\alpha$ containing B. Since each $P \in \mathcal{P}_\alpha$ has semisimple rank 1, this map has projective lines as fibres.
Steinberg and Tits showed that E is a bunch of projective lines — one line of the form $f_\alpha^{-1}(P), P \in \mathcal{P}_\alpha$ for every simple root α — which intersect as prescribed by the edges of Γ_G.

Example: We verify these statements by explicit calculation in the simplest non-trivial case $G = \mathrm{SL}(3, \mathbb{C})$.
As maximal torus T_0 we may take the diagonal matrices and as Borel subgroup the upper triangular matrices.
The root space of $\mathrm{SL}(3, \mathbb{C})$ is



where the character $\alpha_{i,j} \in X^*(T)$ is defined by

$$\alpha_{i,j} \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = a_i - a_j.$$

The eigenspace $\mathfrak{sl}(3, \mathbb{C})_{\alpha_{i,j}}$, for example, consists of those matrices $(a_{i,j})_{i,j=1,\dots,3} \in \mathfrak{sl}(3, \mathbb{C})$ whose single non-zero entry is $a_{2,1}$. Using the exponential map

$$\exp : \mathfrak{sl}(3, \mathbb{C}) \to \mathrm{SL}(3, \mathbb{C})$$

we see that the root subgroup $U_{\alpha_{21}}$ is

$$U_{\alpha_{21}} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \lambda \in \mathbb{C} \right\}.$$

The simple roots are $\alpha_{1,2}$ and $\alpha_{2,3}$ and we obtain

as Dynkin diagram of $\mathrm{SL}(3, \mathbb{C})$ ([13] 12). The unipotent variety $\mathrm{Uni}(\mathrm{SL}(3, \mathbb{C}))$ is given by the unipotent matrices, all regular unipotent elements are conjugate to

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and all subregular unipotent elements are conjugate to

$$x := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have the minimal proper parabolic subgroups

$$P_{\alpha_{1,2}} = \langle B_0, U_{\alpha_{2,1}} \rangle$$

and

$$P_{\alpha_{2,3}} = \langle B_0, U_{\alpha_{3,2}} \rangle$$

which are the stabilizer of the flags

$$0 \subset \mathrm{span}_{\mathbb{C}}\{v_1, v_2\} \subset \mathrm{span}_{\mathbb{C}}\{v_1, v_2, v_3\}$$

and

$$0 \subset \mathrm{span}_{\mathbb{C}}\{v_1\} \subset \mathrm{span}_{\mathbb{C}}\{v_1, v_2, v_3\},$$

respectively. Clearly, $P_{\alpha_{2,3}}$ is conjugate to the parabolic subgroup $P'_{\alpha_{2,3}}$ stabilizing the flag

$$0 \subset \mathrm{span}_{\mathbb{C}}\{v_2\} \subset \mathrm{span}_{\mathbb{C}}\{v_1, v_2, v_3\}.$$

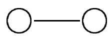
The set of Borel subgroups containing x, which we had identified with E, is given by

$$f^{-1}_{\alpha_{1,2}}(P_{\alpha_{1,2}}) \cup f^{-1}_{\alpha_{2,3}}(P'_{\alpha_{2,3}}).$$

(All the other fibres of the maps $f_{\alpha_{1,2}}$ and $f_{\alpha_{2,3}}$ contain only a finite number of Borel subgroups which contain x.)
The two fibres intersect in a single Borel subgroup: the subgroup stabilizing the maximal flag

$$0 \subset \mathrm{span}_{\mathbb{C}}\{v_2\} \subset \mathrm{span}_{\mathbb{C}}\{v_1, v_2\} \subset \mathrm{span}_{\mathbb{C}}\{v_1, v_2, v_3\}.$$

Hence the Dynkin diagram of E looks like



.

A Appendix - results from Algebraic Geometry

Theorem A.1 (Riemann-Roch on a surface)([17] V.1.6) *If D is any divisor on the non-singular surface X, then*

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2} D \cdot (D - K) + 1 + p_a(X)$$

where K is a canonical divisor *on X ([17] V.1.4.4).*

A simple corollary is the following.

Theorem A.2 (General adjunction formula)([17] Ex. V.1.3.(a)) *If D is an effective divisor on the non-singular surface X, then*

$$2p_a(D) - 2 = D \cdot (D + K).$$

Theorem A.3 (Grothendieck's theorem on formal functions)([16] 4.2.1 or for projective morphisms [17] III.11.1) *Let $f : X \to Y$ be a proper morphism of noetherian schemes and $\mathcal{F}$ a coherent sheaf on X. For $y \in Y$ denote by $m_y \subset \mathcal{O}_{Y,y}$ the maximal ideal of the stalk at y. Then we have a natural isomorphism*

$$0 = \left(\left(R^i f_* \mathcal{F}\right)_y\right)^\wedge \cong \varprojlim_{k=1}^{\infty} H^i\left(f^{-1}(y), \mathcal{F} \otimes_{\mathcal{O}_Y} \frac{\mathcal{O}_{Y,y}}{{m_y}^k}\right) \text{ for all } i \in \mathbb{N}$$

where the completion is taken with respect to the m_y-adic topology.

Theorem A.4 (Castelnuovo's criterion for contracting a curve)([20] IV §15, [17] V.5.7) *If C is a curve on a non-singular surface X with $C \cong \mathbb{P}^1$ and $C^2 = -1$, then there exists a morphism $f : X \to X'$ to a non-singular surface X' which contracts C to a point p, such that X is isomorphic via f to the blow-up of X' with center p, and C is the exceptional curve.*

Theorem A.5 (Zariski's connectedness theorem)([16] 4.3.1, [17] III.11.4) *Let $f : X \to Y$ be a birational morphism between projective varieties and assume that Y is normal. Then f has connected fibers.*

Theorem A.6 (A vanishing theorem of Grothendieck)([17] III.2.7) *For any sheaf of abelian groups $\mathcal{F}$ on a noetherian scheme X of dimension n, we have*

$$H^i(X, \mathcal{F}) = 0 \text{ for } i > n.$$

Theorem A.7 (A vanishing theorem for higher direct image sheaves)([17] III.11.2) *Let $f : X \to Y$ be a projective morphism of noetherian schemes and denote by r the maximal dimension of its fibers. Then for all coherent sheaves $\mathcal{F}$ on X, we have*

$$R^i f_* \mathcal{F} = 0 \text{ for } i > r.$$

References

[1] Artin, M.: *Some numerical criteria for the contractability of curves on algebraic surfaces.* Amer. Journ. of Math. **84** (1962), pp. 485 - 496.

[2] Artin, M.: *On isolated rational singularities of surfaces.* Amer. Journ. of Math. **88** (1966), pp. 129 - 136.

[3] Brieskorn, E.: *Über die Auflösung gewisser Singularitäten von holomorphen Abbildungen.* Math. Ann. **166** (1966), pp. 76 - 102.

[4] Brieskorn, E.: *Rationale Singularitäten komplexer Fächen.* Inventiones math. **4** (1968), pp. 336 - 358.

[5] Brieskorn, E.: *Die Auflösung der rationalen Singularitäten holomorpher Abbildungen.* Math. Ann. **178** (1968), pp. 255 - 270.

[6] Brieskorn, E.: *Singular elements of semisimple algebraic groups.* Actes, Congrès intern. Math. 1970, vol. 2, pp. 279 - 284.

[7] Dimca, A.: *Topics on real and complex singularities.* Vieweg, Braunschweig, 1987.

[8] Demazure, M., H. Pinkham and B. Teissier *Séminaire sur les singularités des surfaces.* Springer, Berlin, 1980.

[9] Durfee, A.: *Fifeteen characterizations of rational double points and simple critical points.* L'Enseignement Mathematique **25** (1979), pp. 131 - 163.

[10] Du Val, P.: *On isolated singularities of surfaces that do not affect the condition of adjunction I, II, III.* Proc. Cambridge Phil. Soc. **30** (1934), pp. 483 - 491.

[11] Du Val, P.: *Homographies, quaternions and rotations.* Clarendon Press, Oxford, 1964.

[12] Fulton, W.: *Introduction to toric varieties.* Princeton Univ. Press, Princeton, 1993.

[13] Fulton, W. and J. Harris: *Representation theory: A first course.* Springer, New York, 1991.

[14] Godement, R.: *Topologie algébrique et théorie des faisceaux.* Herman, Paris, 1958.

[15] Grauert, H.: *Über Modifikationen und exzeptionelle Mengen.* Math. Ann. **146** (1962), pp. 331 - 368.

[16] Grothendieck, A. and J. Dieudonné: *Éléments de géométrie algébrique: vol. 3: Étude cohomologique des faisceaux cohérents.* Institut des Hautes Études, Paris, 1961.

[17] Hartshorne, R.: *Algebraic Geometry.* Springer, New York, 1977.

[18] Kirby, D.: *The structure of an isolated multiple point of a surface I, II, III.* I, II: Proc. London Math. Soc. (3) **6** (1956), pp. 597 - 609. III: London Math. Soc. (3) **7** (1957), pp. 1 - 28.

[19] Klein, F.: *Vorlesung über das Ikosaeder und die Auflösung der Gleichung vom fünften Grade.* Teubner, Leipzig, 1884.

[20] Lamotke, K.: *Regular solids and isolated singularities.* Vieweg, Braunschweig, 1986.

[21] Laufer, H.: *Normal, two-dimensional singularities.* Princeton Univ. Press, Princeton, 1971.

[22] Lipman, J.: *Introduction to resolution of singularities.* Published in: *Algebraic Geometry.* (Proc. Symp. Pure Math. 29, Arcata 1974). AMS, Providence, 1975.

[23] McCleary, J.: *User's guide to spectral sequences.* Publish or Perish, Wilmington, 1985.

[24] Mumford, D.: *The topology of normal singularities of an algebraic surface and a criterion for simplicity.* Publ. Math. Inst. des Hautes Études Sci. **9** (1961), pp. 229 - 246.

[25] Nagata, M.: *Local Rings.* Wiley, New York, 1962.

[26] Prill, D.: *Local classification of quotients of complex manifolds by discontinous groups.* Duke Math. Journ. **34** (1967), p. 375 - 386.

[27] Reid, M.: *Chapters on algebraic surfaces.* Published in: *Complex algebraic varieties,* Kollár, J. (Ed.). AMS, Providence, 1997.

[28] Shatz, S.: *Profinite groups, arithmetic and geometry.* Princeton Univ. Press, Princeton, 1972.

[29] Slodowy, P.: *Simple singularities and simple algebraic groups.* Springer, Berlin, 1980.

[30] Slodowy, P.: *Platonic solids, Kleinian singularities and Lie groups.* Published in: *Algebraic Geometry: Proceedings, Ann Arbour 1981.* Springer, Berlin, 1983.

[31] Tjurina, G.: *On the tautness of rationally contractible curves on a surface.* Izv. Akad. Nauk. USSR Ser. Mat. **32** (1968), pp. 943 - 970; English transl.: Math. USSR Izv. **vol. 2, no. 4** (1968), pp. 907 - 954.

[32] Weibel, C.: *An introduction to homological algebra.* Cambridge Univ. Press, Cambridge, 1994.

[33] Zariski, O. and P. Samuel: *Commutative Algebra, vol. 2.* Van Nostrand, Princeton, 1960.